Computability

Numeric functions

These are the elements of

$$\mathbb{N}^k \longrightarrow \mathbb{N},$$

with $k \geq 0$

Notation $\mathsf{NF}^k = \mathbb{N}^k \longrightarrow \mathbb{N}$

We consider $\mathsf{NF}^0 = \mathbb{N}$

Notation

$$\mathsf{NF} = \bigcup_{k \in \mathbb{N}} \mathsf{NF}^k$$

Well-known functions

$$0 \in \mathsf{NF}^0, \; S \in \mathsf{NF}^1, \; \mathsf{plus} \in \mathsf{NF}^2, \; \mathsf{times} \in \mathsf{NF}^2, \; \Pi_i^k \in \mathsf{NF}^k,$$

with $\Pi_i^k(x_1, \ldots, x_k) = x_i$

Primitive recursive functions

We define subsets $\mathsf{PR}^k \subseteq \mathsf{NF}^k$ and write

$$\mathsf{PR} = \bigcup_k \mathsf{PR}^k$$

the set of *primitive recursive functions* as the least set such that

Initial functions

$0 \in \mathsf{PR}^0$

$S \in \mathsf{PR}^1$

$\Pi_i^k \in \mathsf{PR}^k$

Composition

Let $G_1, \ldots, G_m \in \mathsf{PR}^k$, $H \in \mathsf{PR}^m$; define $F \in \mathsf{NF}^k$ by

$F(\vec{x}) = H(G_1(\vec{x}), \ldots, G_m(\vec{x}))$

then $F \in \mathsf{PR}^k$

Primitive recursion

Let $G \in \mathsf{PR}^k$, $H \in \mathsf{PR}^{k+2}$; define $F \in \mathsf{NF}^{k+1}$ by

$$\begin{aligned} F(0, \vec{x}) &= G(\vec{x}) \\ F(y+1, \vec{x}) &= H(y, F(y, \vec{x}), \vec{x}) \end{aligned}$$

then $F \in \mathsf{PR}^{k+1}$

Primitive recursive relations

Let $R \subseteq \mathbb{N}^k$.

(i) The *characteristic function* of R is $\chi_R \in \mathsf{NF}^k$ defined by

$$\begin{aligned} \chi_R(\vec{x}) &= 0, \quad \text{if } R(\vec{x}), \\ &= 1, \quad \text{else.} \end{aligned}$$

(ii) The relation R is *primitive recursive* if $\chi_R \in \mathsf{PR}^k$

Then we write $R \in \mathsf{PR}^k$.

Developing PR relations

Lemma. Let $R \in \mathsf{PR}$. Then also $\neg R \in \mathsf{PR}^k$.

Proof. $\chi_{\neg R}(\vec{x}) = \overline{sg}(\chi_R(\vec{x}))$. ■

Lemma *Let* $R_1, R_2 \in PR$. *Then also* $R_1 \ \& \ R_2$, $R_1 \vee R_2 \in PR$.

Proof. $\chi_{R_1 \vee R_2}(\vec{x}) = \chi_{R_1}(\vec{x}).\chi_{R_2}(\vec{x})$;

$R_1(\vec{x}) \ \& \ R_2(\vec{x}) = \neg(\neg R_1(\vec{x}) \vee \neg R_2(\vec{x}))$. ■

Developing PR functions

Lemma. *Let* $H_1, H_2, R \in \mathit{PR}^k$*. Define* $F \in \mathit{NF}^k$

$$\begin{aligned} F(\vec{x}) &= H_1(\vec{x}), & \textit{if } R(\vec{x}) \\ &= H_2(\vec{x}), & \textit{else.} \end{aligned}$$

Then $F \in \mathit{PR}^k$.

Proof. $F(\vec{x}) = H_1(\vec{x}).\chi_R(\vec{x}) + H_2(\vec{x}).\chi_{\neg R}(\vec{x})$. ■

Lemma. *Let* $H_1 \in \mathit{PR}^1, G \in \mathit{PR}^2, H \in \mathit{PR}^3$*. Define* $F \in \mathit{NF}^2$ *by*

$$F(x, y) = G(H_1(y), H_2(x, y, x))$$

Then $F \in \mathit{PR}$*.*

Proof. Write $\vec{x} = x, y$. Then

$$\begin{aligned} F(\vec{x}) &= G(H_1(\Pi^2_1(\vec{x})), H_2(\Pi^2_1(\vec{x}), \Pi^2_2(\vec{x}), \Pi^2_1(\vec{x}))) \\ &= G(K_1(\vec{x}), K_2(\vec{x})) \end{aligned}$$

Thus K_1, K_2, and hence also F, are all in PR. ■

Lemma. $Z = (\lambda x.0) \in \mathrm{PR}^1$. Proof. $Z(0) = 0; Z(x+1) = Z(x) = \Pi^2_2(x, Z(x))$. ■

More PR functions

Lemma $+, \times \in \mathrm{PR}^2$.

Proof.
$$\begin{array}{rcl} x+0 & = & x \\ x+(y+1) & = & S(x+y) \end{array}$$
$$\begin{array}{rcl} x.0 & = & 0 \\ x.(y+1) & = & x.y+x \end{array}$$

Lemma. *Let* $G \in PR^{k+1}$. *Define*

$$\begin{array}{rcl} F(\vec{x}, y) & = & \Sigma_{i<y} G(\vec{x}, i) \\ H(\vec{x}, y) & = & \Pi_{i<y} G(\vec{x}, i). \end{array}$$

Then $F, H \in PR^{k+1}$.

Proof. Define
$$\begin{array}{rcl} F(\vec{x}, 0) & = & 0 \\ F(\vec{x}, y+1) & = & F(\vec{x}, y) + G(\vec{x}, y) \end{array}$$
$$\begin{array}{rcl} H(\vec{x}, 0) & = & 1 \\ H(\vec{x}, y+1) & = & H(\vec{x}, y).G(\vec{x}, y). \blacksquare \end{array}$$

More PR relations

Lemma. *Let* $R \in$ *PR*. *Define*

$$\begin{aligned} S(\vec{x}, z) &\Leftrightarrow \exists y{<}z.R(\vec{x}, y) \\ T(\vec{x}, z) &\Leftrightarrow \forall y{<}z.R(\vec{x}, y) \end{aligned}$$

Then $S, T \in$ *PR*.

Proof.
$$\begin{aligned} \chi_S(\vec{x}, z) &= \Pi_{y<z}\chi_R(\vec{x}, y) \\ T(\vec{x}, z) &\Leftrightarrow \neg\exists y{<}z.\neg R(\vec{x}). \ \blacksquare \end{aligned}$$

Lemma. *Let* $R \in PR^{k+1}$. *Define* $F \in NF^k$ *by*

$$\begin{aligned} F(\vec{x}, z) &= \mu y{<}z.R(\vec{x}, y), && \textit{if y exists} \\ &= z, && \textit{else.} \end{aligned}$$

Then $F \in PR^k$.

Proof.
$$\begin{aligned} F(\vec{x}, 0) &= 0 \\ F(\vec{x}, z+1) &= F(\vec{x}, z), && \text{if } \exists y{<}z.R(\vec{x}, y), \\ &= z, && \text{else, if } R(\vec{x}, z), \\ &= z+1, && \text{else. } \blacksquare \end{aligned}$$

Packing and unpacking strings of numbers

There is a bijection $j : \mathbb{N}^2 \to \mathbb{N}$ that is PR with inverses $j_i \in \mathrm{PR}^1$

$$j(j_1(x), j_2(x)) = x$$

[$j(x,y) = \frac{1}{2}(x+y)(x+y+1) + x$ known to Cantor]

There is a bijection $j^m : \mathbb{N}^m \to \mathbb{N}$ that is PR with inverses $j_i^m \in \mathrm{PR}^1$

$$j^m(j_1^m(x), \ldots, j_m^m(x)) = x$$

[Iterate j]

There is a bijection $\langle - \rangle : \bigcup_{k \geq 0} \mathbb{N}^k \to \mathbb{N}$ and a maps $\lambda xj.(x)_j \in \mathrm{PR}^2$, $\mathrm{lh} \in \mathrm{PR}^1$ such that

$$x = \langle\,\rangle \quad \Rightarrow \quad \mathrm{lh}(x) = 0 \;\&\; (x)_i = 0$$
$$x = \langle y_0, \ldots, y_{m-1} \rangle \quad \Rightarrow \quad \mathrm{lh}(x) = m \;\&\; (x)_i = y_i, \qquad \text{for } i < m$$

[$\langle y_0, \ldots, y_{m-1} \rangle = j(m-1, j^m(y_0, \ldots, y_{m-1}))$]

For fixed m one has $(\lambda y_0 \ldots y_{m-1}.\langle y_0, \ldots, y_{m-1} \rangle) \in \mathrm{PR}^m$

There is a map $* \in \mathrm{PR}^2$ such that

$$\langle y_0, \ldots, y_{m-1} \rangle * \langle z_0, \ldots, z_{n-1} \rangle = \langle y_0, \ldots, y_{m-1}, z_0, \ldots, z_{n-1} \rangle$$

The Fibonacci function

Define
$$\begin{aligned} F(0) &= 1 \\ F(1) &= 1 \\ F(n+2) &= F(n) + F(n+1) \end{aligned}$$

Then $F \in \mathsf{PR}^1$.

Proof. Define $H(x) = \langle F(x), F(x+1) \rangle$. Then

$$\begin{aligned} H(0) &= \langle 1, 1 \rangle \\ H(x+1) &= \langle (H(x))_1, (H(x))_0 + (H(x))_1 \rangle \end{aligned}$$

is PR. Hence $F(x) = (H(x))_0$ is PR. ■

Course of value recursion

Given $F \in \mathsf{NF}^{k+1}$ define

$$\overline{F}(\vec{x}, n) = \langle F(\vec{x}, 0), \ldots, F(\vec{x}, n-1) \rangle$$

Given $H \in \mathsf{PR}^{k+2}$. Define

$$F(\vec{x}, n) = H(\overline{F}(\vec{x}, n), \vec{x}, n)$$

Then $F \in \mathsf{PR}$.

Proof. We show first that $\overline{F} \in \mathsf{PR}^{k+1}$. Indeed

$$\begin{aligned} \overline{F}(\vec{x}, 0) &= \langle\,\rangle \\ \overline{F}(\vec{x}, n+1) &= \overline{F}(\vec{x}, n) * \langle F(\vec{x}, n) \rangle \\ &= \overline{F}(\vec{x}, n) * \langle H(\overline{F}(\vec{x}, n), \vec{x}, n) \rangle \end{aligned}$$

Then

$$F(\vec{x}, n) = (\overline{F}(\vec{x}, n+1))_n.$$

Therefore $F \in \mathsf{PR}$. ■

Partial recursive functions

A *partial function* $\psi : \mathbb{N}^k \rightharpoonup \mathbb{N}$ is a relation $\psi \subseteq \mathbb{N}^k \times \mathbb{N}$ such that

$$(x, y), (x, y') \in \psi \;\Rightarrow\; y = y'$$

If $(x, y) \in \psi$, then we write $\psi(x)\downarrow$ ($\psi(x)$ *is defined*) and $\psi(x) \simeq y$.

If for no y one has $(x, y) \in \psi$, then we write $\psi(x)\uparrow$ ($\psi(x)$ *is undefined*)

Initial functions	$0 \in \mathcal{PR}^0$
	$S \in \mathcal{PR}^1$
	$\Pi_i^k \in \mathcal{PR}^k$
Composition	Let $\psi_1, \ldots, \psi_m \in \mathcal{PR}^k$, $\chi \in \mathcal{PR}^m$; define $\varphi \in \mathsf{NF}^k$ by
	$\varphi(\vec{x}) \simeq \chi(\psi_1(\vec{x}), \ldots, \psi_m(\vec{x}))$
	then $\varphi \in \mathcal{PR}^k$
Primitive recursion	Let $\psi \in \mathcal{PR}^k$, $\chi \in \mathcal{PR}^{k+2}$; define $\varphi \in \mathsf{NF}^{k+1}$ by
	$\varphi(0, \vec{x}) \simeq \psi(\vec{x})$ $\varphi(y+1, \vec{x}) \simeq \chi(y, \varphi(y, \vec{x}), \vec{x})$ then $\varphi \in \mathcal{PR}^{k+1}$
Minimalization	Let $\chi \in \mathcal{PR}^{k+1}$ and define $\varphi(\vec{x}) \simeq \mu y.[\chi(\vec{x}, y) \simeq 0]$ then $\varphi \in \mathcal{PR}^k$

Small print about equality

In an equation

$$t \simeq s$$

with expressions that are partially defined, the meaning is

$$[t\downarrow \;\Rightarrow\; [s\downarrow \,\&\, t = s]] \,\&\, [s\downarrow \;\Rightarrow\; [t\downarrow \,\&\, t = s]]$$

We understand that

$$\begin{aligned}\chi(\psi_1(\vec{x}), \ldots, \psi_k(\vec{x}))\downarrow \quad &\Leftrightarrow \quad \exists y_1, \ldots, y_k. \\ &\qquad [\psi_1(\vec{x}) = y_1 \;\&\; \ldots \;\&\; \psi_k(\vec{x}) = y_k \\ &\qquad \;\&\; \chi(y_1, \ldots, y_k)\downarrow]\end{aligned}$$

$$\begin{aligned}\mu y.[\chi(\vec{x}, y) \simeq 0]\downarrow \quad &\Leftrightarrow \quad \exists y.[\chi(\vec{x}, y) \simeq 0 \;\& \\ &\qquad \forall z{<}y.\chi(\vec{x}, z)\downarrow \;\&\; \chi(\vec{x}, z) \not\simeq 0]\end{aligned}$$