First-order languages and logic

Structures, signatures, and languages

A ring $\mathcal{R} = \langle R, +, \cdot, 0, 1\rangle$

Signature $\langle 2, 2, 0, 0\rangle$: arity of the functions, constants as 0-ary fcts

An ordered field $\mathcal{K} = \langle K, +, \cdot, -, ^{-1}, 0, 1, <\rangle$

Signature $\langle 2, 2, 1, 1, 0, 0; 2\rangle$: $-, ^{-1}$ as unary operations

Language for $\mathcal{R}$: $\langle f_+, f_\cdot, c_0, c_1\rangle$

Language for $\mathcal{K}$: $\langle f_+, f_\cdot, f_-, f_{-1}, c_0, c_1, \underset{\cdot}{<}\rangle$ or $\langle f_+, f_\cdot, f_-, f_{-1}, c_0, c_1, R\rangle$

each function- or relation-symbol comes with an arity

Abstract syntax of first order L

variables	V	:=	x | V′	
terms	t	:=	V | f(t_1,...,t_k)	(arity of f is k)
atoms	A	:=	⊥ | t=t | R(t_1,...,t_k)	(arity of R is k)
formulas	F	:=	A | F→F | F∨F | F∧F | ∀V F | ∃V F	

Interpretation of terms

$\mathcal{S} = \langle X, F_1, \ldots, F_n, P_1, \ldots, P_m \rangle$ with given arities

symbols in language $\langle f_1, \ldots, f_n, R_1, \ldots, R_m \rangle$

Interpretation of terms t in $\mathcal{S}$ depending on a valuation $\rho : V \to X$

t	$(t)^{\mathcal{S}}_{\rho}$
x	$\rho(x)$
$f(t_1, \ldots, t_k)$	$F((t_1)^{\mathcal{S}}_{\rho}, \ldots, (t_k)^{\mathcal{S}}_{\rho})$

Satisfaction

Satisfaction depending on ρ of a formula in $\mathcal{S}$

$$
\begin{array}{rcl}
\mathcal{S}, \rho \models \bot & \Leftrightarrow & \text{never} \\
\mathcal{S}, \rho \models t_1 = t_2 & \Leftrightarrow & (t_1)^{\mathcal{S}}_{\rho} = (t_2)^{\mathcal{S}}_{\rho} \\
\mathcal{S}, \rho \models A_1 \vee A_2 & \Leftrightarrow & \mathcal{S}, \rho \models A_1 \text{ or } \mathcal{S}, \rho \models A_2 \\
\mathcal{S}, \rho \models A_1 \wedge A_2 & \Leftrightarrow & \mathcal{S}, \rho \models A_1 \text{ and } \mathcal{S}, \rho \models A_2 \\
\mathcal{S}, \rho \models A_1 \rightarrow A_2 & \Leftrightarrow & \mathcal{S}, \rho \models A_1 \text{ implies } \mathcal{S}, \rho \models A_2 \\
\mathcal{S}, \rho \models \forall x A & \Leftrightarrow & \mathcal{S}, \rho[x := a] \models A, \text{ for all } a \in X \\
\mathcal{S}, \rho \models \exists x A & \Leftrightarrow & \mathcal{S}, \rho[x := a] \models A, \text{ for some } a \in X
\end{array}
$$

Defining $\neg F := (F \rightarrow \bot)$ we get

$$\mathcal{S}, \rho \models \neg F \;\Leftrightarrow\; \mathcal{S}, \rho \not\models F$$

Consistency, completeness

$$\mathcal{S} \models A \quad \Leftrightarrow \quad \text{for all } \rho[\mathcal{S}, \rho \models A]$$

Γ stands for a set of formulas

$$\mathcal{S} \models \Gamma \quad \Leftrightarrow \quad \text{for all } A{\in}\Gamma[\mathcal{S} \models A]$$

$$\Gamma \models A \quad \Leftrightarrow \quad \text{for all } \mathcal{S}[\mathcal{S} \models \Gamma \ \Rightarrow \ \mathcal{S} \models A]$$

Γ is inconsistent iff $\Gamma \models A \;\&\; \Gamma \models \neg A$, for some $A{\in}F$

Γ is complete iff $\Gamma \models A$ or $\Gamma \models \neg A$, for all $A{\in}F$

Propositional logic

→-E $$\frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}$$

→-I $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

∧-E $$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

∧-I $$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

∨-E $$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma \vdash C}{\Gamma \vdash C}$$

∨-I $$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\frac{A \in \Gamma}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$ (classical logic)

Abbreviations

$\Gamma \vdash A$ stands for $\emptyset \vdash A$

$\neg A$ stands for $A \to \bot$

$A \leftrightarrow B$ stands for $(A \to B) \wedge (B \to A)$

Admissible rules

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \bot} \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash B \to A} \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash B \to A}{\Gamma \vdash A \leftrightarrow B}$$

Predicate logic

$$\forall\text{-E} \quad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t)} \qquad \forall\text{-I} \quad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A(x)}\ x \notin \mathrm{FV}(\Gamma)$$

$$\exists\text{-E} \quad \frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(x) \vdash B}{\Gamma \vdash B} \qquad \exists\text{-I} \quad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}$$

Soundness and completeness

$$\Gamma \vdash A \quad \Rightarrow \quad \Gamma \models A, \qquad \text{soundness}$$

$$\Gamma \models A \quad \Rightarrow \quad \Gamma \vdash A, \qquad \text{completeness (Gödel [1930])}$$

Exercises

$\vdash A \to A$

$\vdash (A \to (A \to B)) \to (A \to B)$

$\vdash \neg\neg(A \vee \neg A)$

$\vdash \neg\exists x A(x) \leftrightarrow \forall x \neg A(x)$

$\vdash \neg\forall x A(x) \leftrightarrow \exists x A(x)$

Show that $\Gamma \vdash A$ is equivalent to the notion as in lecture notes