Doing it yourself

Hofstadter's MU puzzle

Consider the alphabet $\Sigma = \{\mathsf{M}, \mathsf{I}, \mathsf{U}\}$.

For a string $w \in \Sigma^*$ we define when it is *derivable*, notation $\vdash w$

axiom		MI	
rules	$w\mathsf{I}$	$\Rightarrow$	$w\mathsf{IU}$
	$\mathsf{M}w$	$\Rightarrow$	$\mathsf{M}ww$
	$w\mathsf{III}v$	$\Rightarrow$	$w\mathsf{U}v$
	$w\mathsf{UU}v$	$\Rightarrow$	wv

1. Give four examples of $w \in \Sigma^*$ such that $\vdash w$
2. Give four non-examples
3. Show that $\nvdash \mathsf{MU}$

Coding words

Define a map $\ulcorner\ \urcorner : \Sigma^* \mapsto \mathbb{N}$ such that

1. There is a primitive recursive function f satisfying

$$\ulcorner wv \urcorner = f(\ulcorner w \urcorner, \ulcorner v \urcorner)$$

2. There is a primitive recursive function g such that

$$\begin{aligned} g(\ulcorner w \urcorner) &= 0, && \text{if } w = \lambda \text{, the empty string} \\ &= 1, && \text{if } w \in \Sigma \text{, a single letter} \\ &= 2, && \text{if } w = uv \text{, for some } u, v \in \Sigma^*,\ u, v \neq \lambda \end{aligned}$$

3. There are primitive recursive functions h_1, h_2 such that for $s \in \Sigma$

$$h_1(\ulcorner sw \urcorner) = \ulcorner s \urcorner$$

$$h_2(\ulcorner sw \urcorner) = \ulcorner w \urcorner$$

Coding derivations and ‘theorems’

Show that there is a Σ_1 formula $B(x)$ of PA such that

$$\vdash w \quad \Rightarrow \quad \mathsf{PA} \vdash B(\overline{w})$$

$$\mathsf{PA} \vdash \neg U$$

$$\mathsf{PA} \vdash \neg B(\overline{\mathsf{MU}})$$

where $\overline{w} = \overline{\ulcorner w \urcorner}$

[Hint. Do what is stated on the first line]