Course Lambda calculus: test 2.4.2012.

1. For a lambda term M define the directed graph

$$G_\beta(M) = \{N \mid M \twoheadrightarrow_\beta N\}, \text{ directed by } \to_\beta.$$

   Draw $G_\beta(M)$ for $M = \mathbf{c}_2\mathsf{IK}$, where $\mathbf{c}_2 \triangleq \lambda fx.f(fx)$.
   [NB. In a directed graph, there may be two arcs from one node to one other node.]

2. (a) Show that there exists an $F \in \Lambda$ such that for all $M \in \Lambda$ one has

$$FM = \mathsf{S}(FM)(FM)$$

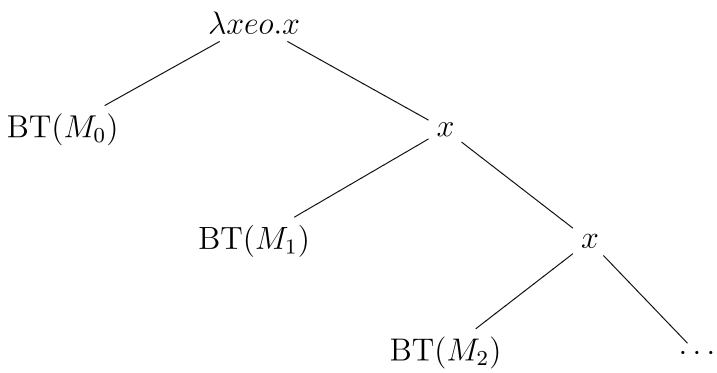
   (b) Idem with

$$FM = \mathsf{S}(\ulcorner F \urcorner M)(\ulcorner F \urcorner M)$$

   (c) Idem with

$$FM = \mathsf{S}(FM)(\ulcorner F \urcorner M)$$

3. Write down a closed lambda term M such that $\mathrm{BT}(M)$ is as follows.



,

   where

$$M_n = \begin{cases} e & \text{if } n \text{ is even;} \\ o & \text{if } n \text{ is odd.} \end{cases}$$

4. Extended λP with a new type constructor Σ with formation rule

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x{:}A \vdash B : *}{\Gamma \vdash \Sigma x{:}A.B : *}$$

Add the following typing rules (motivated by the logical rules for the existential quantifier):

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[x{:=}a]}{\Gamma \vdash \epsilon ab : \Sigma x{:}A.B}\ \Sigma\text{-intro}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_1 M : A}\ \Sigma\text{-elim1}$$

$$\frac{\Gamma \vdash M : \Sigma x{:}A.B}{\Gamma \vdash \pi_2 M : B[x{:=}\pi_1 M]}\ \Sigma\text{-elim2}$$

Finally, add to to β-reduction the following computation rule:

$$\begin{array}{lll} \pi_1(\epsilon ab) = a & : & A \\ \pi_2(\epsilon ab) = b & : & B[x{:=}a] = B[x{:=}\pi_1(\epsilon ab)]. \end{array}$$

Prove the *Constructive Axiom of Choice* by exhibiting a term AC such that, with $\Gamma = \{A{:}*, B{:}*, P : A \to B \to *\}$, one has

$$\Gamma \vdash \mathrm{AC} : (\Pi x{:}A.\Sigma y{:}B.Pxy) \to (\Sigma f{:}A{\to}B.\Pi x{:}A.Px(fx)).$$

5. In algebra one has $x^{m+n} = x^m x^n$.

   In the polymorphic system $\lambda 2$, define

$$\begin{aligned} A + B &= \forall \gamma{:}*.(A \to \gamma) \to (B \to \gamma) \to \gamma; \\ A \times B &= \forall \gamma{:}*.(A \to B \to \gamma) \to \gamma; \\ A \Leftrightarrow B &= (A \to B) \times (B \to A). \end{aligned}$$

   Give a term inhabiting

$$((A + B) \to C) \ \Leftrightarrow\ ((A \to C) \times (B \to C)).$$