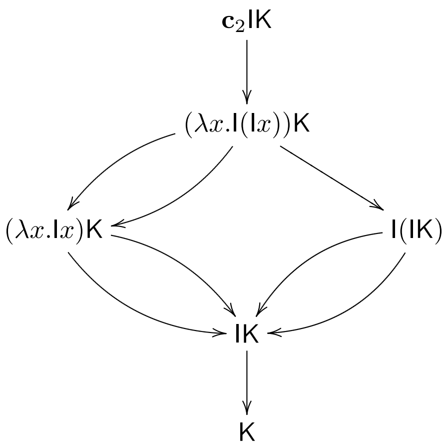
Lambda Calculus: answers to the test problems of April 2, 2012.

1. Draw the reduction graph $G_\beta(\mathbf{c}_2\mathsf{IK})$.
   The collection of nodes is

$$\{(\lambda fx.f(fx))\mathsf{IK}, (\lambda x.\mathsf{I}(\mathsf{I}x))\mathsf{K}, (\lambda x.\mathsf{I}x)\mathsf{K}, \mathsf{I}(\mathsf{IK}), \mathsf{IK}, \mathsf{K}\}.$$

   The directed graph is



2. (a) Show that there is a term $F \in \Lambda$ such that for all $M \in \Lambda$ one has

$$FM = \mathsf{S}(FM)(FM)$$

   This is implied by

$$\begin{aligned} F &= \lambda m.\mathsf{S}(Fm)(Fm) \\ &= (\lambda fm.\mathsf{S}(fm)(fm))F \\ &\triangleq GF \end{aligned}$$

   Therefore we can take the fixed point $F = \mathsf{Y}G$.

   (b) Now we want an F such that for all M

$$FM = \mathsf{S}(\ulcorner F \urcorner M)(\ulcorner F \urcorner M)$$

   Now we can take $F \triangleq \mathsf{Y}_2\ulcorner G \urcorner$. Indeed, according to the second fixed point theorem $F = G\ulcorner F \urcorner = \lambda m.\mathsf{S}(\ulcorner F \urcorner m)(\ulcorner F \urcorner m)$, so F works.

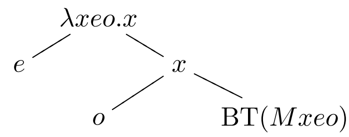
   (c) Finally we want a term F such that $FM = \mathsf{S}(FM)(\ulcorner F \urcorner M)$.
   This is implied by

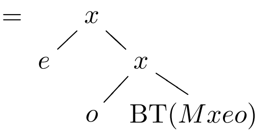
$$F = \lambda m.\mathsf{S}(\mathsf{E}\ulcorner F \urcorner M)(\ulcorner F \urcorner M) = (\lambda fm.\mathsf{S}(\mathsf{E}fM)(fM))\ulcorner F \urcorner,$$

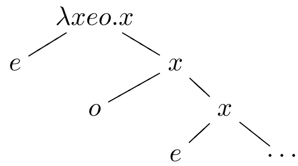
   so we can apply the second fixed point theorem again

$$F \triangleq \mathsf{Y}_2\ulcorner \lambda fm.\mathsf{S}(\mathsf{E}fM)(fM) \urcorner.$$

3. Define $M \triangleq (\lambda z.(\lambda xeo.xe(xo(zzxeo))))(\lambda z.(\lambda xeo.xe(xo(zzxeo))))$. Then $M \twoheadrightarrow_\beta (\lambda xeo.xe(xo(Mxeo)))$ and

   BT(M) = 

   , BT$(Mxeo)$ = 

   Hence BT(M) = 

   .

4. Given a term H of type $\Pi x{:}A.\, \Sigma y{:}B.\, Pxy$, we can define a function term $f : A \to B$ by composing H with the first projection:

$$f = \lambda x{:}A.\pi_1(Hx) : A \to B$$

   For the second component, we need a term of type

$$\Pi x{:}A.\, B(fx) \tag{1}$$

   Composing H with the second projection gives us the term

$$F = \lambda x{:}A.\pi_2(Hx) : \Pi x{:}A.\, B(\pi_1(Hx)) \tag{2}$$

   The types in (1) and (2) are not the same, but they are convertible. Indeed,

$$fx = (\lambda x{:}A.\pi_1(Hx))x =_\beta \pi_1(Hx)$$

   So one application of the conversion rule yields that $F : (1)$. Now

$$\epsilon f F : \Sigma_{f:A\to B}\Pi_{x:A}B(fx)$$

5. Introduce the following notation:

$$\begin{array}{rcl}
\mathrm{in}_L\, a & := & \lambda\gamma : *.\, \lambda f : A \to \gamma.\, \lambda g : B \to \gamma.\, fa \\
\mathrm{in}_R\, b & := & \lambda\gamma : *.\, \lambda f : A \to \gamma.\, \lambda g : B \to \gamma.\, gb \\
\langle a, b\rangle & := & \lambda\gamma : *.\, \lambda f : A \to B \to \gamma.\, fab \\
\pi_1\, p & := & pA(\lambda x : A.\, \lambda y : B.\, x) \\
\pi_2\, p & := & pA(\lambda x : B.\, \lambda y : B.\, y)
\end{array}$$

   We can do the two implications separately:

$$\begin{array}{rcl}
F_1 & = & \lambda f : A + B \to C.\, \langle \lambda x : A.\, f\,(\mathrm{in}_L\, x), \lambda y : B.\, f\,(\mathrm{in}_R\, y)\rangle \\
 & & : (A + B \to C) \to (A \to C) \times (B \to C) \\
F_2 & = & \lambda p : (A \to C) \times (B \to C).\, \lambda x : A + B.\, xC(\pi_1\, p)(\pi_2 p) \\
 & & : (A \to C) \times (B \to C) \to (A + B \to C)
\end{array}$$

   The solution is $\langle F_1, F_2\rangle$.