Research-problems

1. Axiomatizing coding (Smullyan: Diagonalization and Self-Reference, Oxford Science Publications, 1994).

   (1) The theory of combinators CL has terms defined by the following abstract syntax

$$\begin{array}{rcll} V & := & x \mid V' & \text{(variables)} \\ t & := & V \mid \mathsf{K} \mid \mathsf{S} \mid t_1 t_2 & \text{(terms)} \end{array}$$

   The theory is axiomatized by (the universal closure of)

$$\begin{array}{rcl} \mathsf{K}xy & = & x; \\ \mathsf{S}xyz & = & xz(yz). \end{array}$$

   A closed term is a term without any variables. $\mathsf{CL}^\circ$is the set of closed terms.

   (2) Show that CL is undecidable (Grzegorczyk).

   (3) A *coding* is a map $\ulcorner\ \urcorner : \mathsf{CL}^\circ \to \mathsf{CL}^\circ$. (Should we require that $\ulcorner P \urcorner$ is in 'normal form'?)

   (4) A coding can have several properties.

$$\begin{array}{lll} (\mathsf{A}) & \exists \mathsf{A} \in \mathsf{CL}^\circ \forall P, Q \in \mathsf{CL}^\circ. & \mathsf{A}\ulcorner P\urcorner\ulcorner Q\urcorner = \ulcorner PQ\urcorner; \\ (\mathsf{N}) & \exists \mathsf{N} \in \mathsf{CL}^\circ \forall P \in \mathsf{CL}^\circ. & \mathsf{N}\ulcorner P\urcorner = \ulcorner\ulcorner P\urcorner\urcorner; \\ (\mathsf{Z}) & \exists \mathsf{Z}_\mathsf{K}, \mathsf{Z}_\mathsf{S} \in \mathsf{CL}^\circ. & \left\{\begin{array}{rcll} \mathsf{Z}_\mathsf{K} P & = & \mathsf{K}, & \text{if } P \equiv \mathsf{K}, \\ & = & \mathsf{K}(\mathsf{SKK}), & \text{else;} \\ \mathsf{Z}_\mathsf{S} P & = & \mathsf{K}, & \text{if } P \equiv \mathsf{S}, \\ & = & \mathsf{K}(\mathsf{SKK}), & \text{else;} \end{array}\right. \\ (\mathsf{P}) & \exists \mathsf{P}_1, \mathsf{P}_2 \in \mathsf{CL}^\circ. & \left\{\begin{array}{rcll} \mathsf{P}_1 P & = & \mathsf{K}, & \text{if } P \equiv \mathsf{K}, \\ & = & \mathsf{K}(\mathsf{SKK}), & \text{else;} \\ \mathsf{Z}_\mathsf{S} P & = & \mathsf{K}, & \text{if } P \equiv \mathsf{S}, \\ & = & \mathsf{K}(\mathsf{SKK}), & \text{else;} \end{array}\right. \\ (\mathsf{A}^2) & \exists \mathsf{A}^2 \in \mathsf{CL}^\circ. & \mathsf{A}^2\ulcorner\ulcorner P\urcorner\urcorner\ulcorner\ulcorner Q\urcorner\urcorner = \ulcorner\ulcorner PQ\urcorner\urcorner; \\ (\mathsf{A}_k) & \exists \mathsf{A}_k \in \mathsf{CL}^\circ. & \mathsf{A}_k\ulcorner P\urcorner \ldots \ulcorner P_k\urcorner = \ulcorner\ulcorner P_1 \ldots P_k\urcorner\urcorner; \\ (\mathsf{E}) & \exists \mathsf{E} \in \mathsf{CL}^\circ. & \mathsf{E}\ulcorner P\urcorner = P; \\ (\mathsf{Y}_2) & \exists \mathsf{Y}_2 \in \mathsf{CL}^\circ. & \mathsf{Y}_2\ulcorner F\urcorner = F\ulcorner \mathsf{Y}_2\ulcorner F\urcorner\urcorner; \end{array}$$

   (5) A coding is *admissible* iff A, N hold; it is *pre-admissible* iff Z, P hold.

(6) Show that in a pre-admissible coding one has E. [Hint. Establish an appropriate recursion principle.]

(7) Show that in an admissible coding one has A_k and Y_2.

(8) Show that in an admissible coding satisfying E one has A^2.

(9) Show that in a pre-admissible coding satisfying A^2 one has N.

(10) Research question. Does E hold in every admissible coding?

(11) Research question. Is every admissible coding pre-admissible?

(12) Research question. Is every pre-admissible coding admissible?

2. Naive terms. An untyped lambda term M is called *naive* if for every redex $(\lambda x.P)Q$ subterm of M one has that

$$P(x := Q) \equiv_\alpha P[x := Q].$$

Here $P(x := Q)$ is *naive substitution*

P	$P(x := Q)$
x	Q
y	$y\ (y \neq x)$
$(P_1 P_2)$	$(P_1(x := Q))(P_2(x := Q))$
$(\lambda z.P)(x := Q)$	$\lambda z.(P(x := Q))$

Moreover $M \equiv_\alpha N$ if N results from M by a number of changes of names of bound variables, axiomatized by

$$M \equiv_\alpha M[x := y], \text{ with } y \notin \mathrm{FV}(M) \cup \mathrm{BV}(M)$$

and

M	$\mathrm{FV}(M)$	$\mathrm{BV}(M)$
x	$\{x\}$	$\emptyset$
(PQ)	$\mathrm{FV}(P) \cup \mathrm{FV}(Q)$	$\mathrm{BV}(P) \cup \mathrm{BV}(Q)$
$(\lambda x.P)$	$\mathrm{FV}(P)/\{x\}$	$\mathrm{BV}(P) \cup \{x\}$.

Notice that $x(\lambda yz.y)z$ is naive, but $(\lambda yz.y)z$ is not naive. Let M be an untyped term. Define

M is *hereditarily naive* $\iff \forall N.[M \twoheadrightarrow_{\boldsymbol{\beta}} N \;\Rightarrow\; N$ naive$]$.

For example $\Omega \equiv \omega\omega$ with $\omega \equiv (\lambda x.xx)$ is heriditarily naive but $\omega \mathbf{c}_1$ is not. Indeed,

$$\begin{aligned} \omega \mathbf{c}_1 &= \mathbf{c}_1 \mathbf{c}_1 \\ &= (\lambda f x.fx)\mathbf{c}_1 \\ &= \lambda x.\mathbf{c}_1 x \equiv \lambda x(\lambda f x.fx)x \\ &= \lambda x \lambda x'.xx' \equiv_\alpha \mathbf{c}_1 \not\equiv_\alpha \lambda x.(\lambda x.xx) \equiv \lambda x.\omega! \end{aligned}$$

Problem. Characterize the notion of hereditary naivity. Is it decidable?