Lambda Calculus, Week 3

In-class problems

1. Let $F, G \in \Lambda^{\emptyset}$. Show that if $Fx = Gx$, then for all $M \in \Lambda$ one has $FM = GM$. The assumption that F, G are closed is necessary.

2. Write down a lambda term $F \in \Lambda^{\emptyset}$ such that

$$F\ulcorner xy \urcorner = x,$$

where we use the encoding of lambda terms by Mogensen.

3. Let $W \in \Lambda^{\emptyset}$. Show that there are $M, F \in \Lambda^{\emptyset}$ such that

$$\begin{array}{rcl} FM & = & MW \\ Mf & = & f\ulcorner M \urcorner \end{array}$$

Conclude $FM = W\ulcorner M \urcorner$.

4. For a given T there exists a program P such that

$$\begin{array}{rcll} P\mathbf{c}_k & = & \mathbf{c}_{k+1}, & \text{if } k \text{ is even,} \\ & = & T\ulcorner P \urcorner \mathbf{c}_k, & \text{otherwise.} \end{array}$$

Take-home problems

A finite list $M_{n_1}, \ldots, M_0 \in \Lambda$ is represented in Λ by $[M_{n_1}, \ldots, M_0] \in \Lambda$ defined as follows.

$$\begin{array}{rcl} [\,] & \triangleq & \text{false}\ ; \\ [M_n, M_{n-1}, \ldots, M_0] & \triangleq & \pi M_n [M_{n-1}, \ldots] \end{array}$$

Then

$$[M_0, \ldots, M_n] := \langle M_0, \langle M_1, \langle M_2, \cdots \langle M_n, \mathsf{F} \rangle \cdots \rangle \rangle \rangle$$

We can also represent an infinite *stream* of terms $\{M_0, M_1, \ldots\}$ by a term that looks like

$$[M] = \langle M_0, \langle M_1, \cdots \rangle \rangle$$

In this case, the term $[M]$ is called the *uniform enumeration* of $\langle M_n \rangle$.

For example, using the fixed-point theorem, we can define a term

$$\mathbb{N}x = \texttt{cons}\ x\ (\mathbb{N}(S^+x))$$

Where S^+ is the successor on Church numerals. Then $[M] = \mathbb{N}\mathbf{c}_0$ is a uniform enumeration of $\{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \ldots\}$.

1. Given n, define a term P_n such that

$$\mathsf{P}_n[M] = M_n$$

for any uniform enumeration $[M]$.

[Hint. It could be easier to solve 2 first.]

2. Define a term P such that

$$\mathsf{P}\mathbf{c}_n[M] = M_n$$

3. Given a term N, define a term $@_N$ such that, for any uniform enumeration $\langle M_n \rangle$, $@_N$ is a uniform enumeration of $\langle M_n N \rangle$.

Then define a term $@$ such that $@_N = @N$ for each N.

4. Define a term $\texttt{zip}$, such that

$$\texttt{zip}[M][N] = \langle M_0, \langle N_0, \langle M_1, \langle N_1, \ldots \rangle\rangle\rangle\rangle$$

is a uniform enumeration of $\{M_0, N_0, M_1, N_1, \ldots\}$.

5. If for every fixed n, the sequence $\{M_{n,m}\}_m$ is uniformly enumerated by $[M_n]$, and the sequence of terms $\{[M_n]\}_n$ is uniformly enumerated by $[M]$, define a term $[M^+]$ which uniformly enumerates the countable set $\{M_{n,m}\}$ (using any ordering you wish).

6. Use the fixed-point theorem to define a uniform enumeration $\mathbb{C}$ of the set of combinators $\mathcal{C}$. Prove that every combinator indeed occurs in the stream enumerated by $\mathbb{C}$.

[Hint. Take $\mathbb{C} = [\mathsf{I}, \mathsf{K}, \mathsf{S}, \_\_\_\_]$, where "____" depends on $\mathbb{C}$.]

7. Give a term E with the following property: For every *closed* lambda term $t \in \Lambda^0$, there exists an n such that

$$E\mathbf{c}_n = t$$