1 Week 1

In-class problems

1. Let $\mathsf{S} := \lambda xyz.xz(yz)$. Show carefully that $\mathsf{S}XYZ \twoheadrightarrow_\beta XZ(YZ)$.

2. For $M \in \Lambda$ its *reduction graph* is the di-graph

$$\mathcal{G}(M) = (G, E)$$

where
$$\begin{array}{rcl} G &=& \{N \in \Lambda \mid M \twoheadrightarrow_\beta N\} \\ E &=& \{(M,N) \in G^2 \mid M \to_\beta N\}. \end{array}$$

Draw $\mathcal{G}(M)$ for $M = WWW$, where $W = \lambda xy.xyy$.
[Hint. It can be counted by the fingers of one hand.]

3. (a) Define the lambda term $\pi := \lambda xyf.fxy$. Use the notation

$$\langle M, N \rangle := \pi\, MN;$$

it 'packages' two lambda terms in one single lambda term. Show that there are $\pi_1, \pi_2 \in \Lambda$ such that

$$\begin{array}{c} \pi_1\langle M, N\rangle \twoheadrightarrow_\beta M, \\ \pi_2\langle M, N\rangle \twoheadrightarrow_\beta N. \end{array}$$

(b) Show that for $F, G \in \Lambda$ there exists $F^\wedge, G^\vee \in \Lambda$ such that

$$\begin{array}{rcl} F^\wedge\langle x, y\rangle &=& Fxy, \\ G^\vee xy &=& G\langle x, y\rangle. \end{array}$$

(c) Show that there are $T_{\mathtt{currying}}, T_{\mathtt{uncurrying}} \in \Lambda$ such that

$$\begin{array}{rcl} T_{\mathtt{currying}}\, F &=& F^\wedge, \\ T_{\mathtt{uncurrying}}\, G &=& G^\vee. \end{array}$$

(d) Check whether
$$\begin{array}{rcl} T_{\mathsf{uncurry}}\,(T_{\mathsf{curry}} f) &\twoheadrightarrow_\beta& f, \\ T_{\mathsf{curry}}\,(T_{\mathsf{uncurry}} f) &\twoheadrightarrow_\beta& f. \end{array}$$

4. Evaluate (according to the intuitive meaning of lambda abstraction)

$$\left(\lambda f.\int_0^1 (f \circ (\lambda y.e^y))'(x)dx\right)(\lambda x.x^2).$$

Take-home problems

1. Construct a CL-term O such that $OP =_{\mathsf{CL}} O$ (the *Ogre*).

2. A λ-term M is called in *normal form* (nf) if for no N one has $M \to_\beta N$. You may use the fact that M has at most one nf. If it exists we denote it by $\mathsf{nf}(M)$.

   Let the *length* $|M|$ of a term M be the number of symbols in M not counting lambdas and parentheses.

   (a) Write a term t such that $|t| \leq 40$ and $|\mathsf{nf}(t)| > 4000$. [Hint. For $n \in \mathbb{N}$ and $F, M \in \Lambda$ define $\mathbf{c}_n := \lambda f x. f^n x$, where $F^n M$ as defined as follows

   $$\begin{array}{rcl} F^0 M & := & M \\ F^{n+1} M & := & F(F^n M). \end{array}$$

   Show that $\mathbf{c}_n \mathbf{c}_m =_\beta \mathbf{c}_{m^n}$.]

   (b) Write a term t such that $|t| \leq 30$ and $|\mathsf{nf}(t)| > 10^{10^{10^{10^{10}}}}$.

3. Draw $\mathcal{G}(M)$ for the following M.

   (i) $M = VV$, where $V = \lambda x.\mathsf{I}xx$.

   (ii) $M = UU$, where $U = \lambda x.\mathsf{I}(xx)$.

   [Hint. One can be counted using the fingers of two hands, the other cannot be counted using all the fingers of all primates on Earth.]

4. (a) Prove that there is no "Dirac delta" in the lambda calculus:

   $$\delta_{\mathsf{I}} M = \begin{cases} \lambda xy.x & M = \mathsf{I} \\ \lambda xy.y & \text{otherwise} \end{cases}$$

   (b) Let $\emptyset \neq \mathcal{F} \neq \Lambda^0$ be a set of terms closed under (beta) equality:

   $$M \in \mathcal{F},\ M = N \Longrightarrow N \in \mathcal{F}$$

   Prove that there is no term $\delta_{\mathcal{F}}$ such that

   $$\delta_{\mathcal{F}} M = \begin{cases} \lambda xy.x & M \in \mathcal{F} \\ \lambda xy.y & \text{otherwise} \end{cases}$$

   (This is a weak form of Scott's Theorem.)