Lambda Calculus

Week 3

Solvability and Böhm-trees

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Head normal forms

Prop. Every λ-term M is of one of the following shapes

$$M \equiv \lambda\vec{x}.yN_1 \ldots N_n$$

$$M \equiv \lambda\vec{x}.(\lambda y.P)QN_1 \ldots N_n$$

Proof. Induction on the structure of M. ■

Def. In the first case M is called in *head normal form* (hnf)

and y is the *head variable*

In the second case $(\lambda y.P)Q$ is called the *head redex*

Def. A term M *has a hnf* if $M =_\beta N$ for some hnf N

For example IK is not an hnf but has the hnf K

$\Delta\Delta$ has no hnf (nor a nf)

Y has no nf, but an hnf $\lambda f.f(\omega_f\omega_f)$

Propositions about hnfs

Prop. Given M continued contraction of the head-redex will find the hnf of M if it exists (otherwise continue forever)

Def. The hnf found this way is called the *principal hnf (phnf)*

Example. $\lambda x.\mathsf{I}x(\mathsf{IK})$ has as hnf $\lambda x.x\mathsf{K}$, but the phnf $\lambda x.x(\mathsf{IK})$

Solvable terms

Def. (i) Let $M \in \Lambda^{\emptyset}$. Then M is *solvable* iff $\exists n \in \mathbb{N}\ \exists N_1 \ldots N_n . M\vec{N} = \mathsf{I}$

(ii) Let $M \in \Lambda$ with $\mathrm{FV}(M) = \{\vec{x}\}$. Then M is *solvable* if its *closure* $\lambda\vec{x}.M$ is solvable.

Examples. (i) x, $x\mathsf{K}$, $x(\Delta\Delta)$, $\langle\Delta\Delta, \mathsf{S}\rangle$, $\langle\Delta\Delta, \Delta\Delta\rangle$, and Y are all solvable

(ii) $\Delta\Delta$, YK, are unsolvable

Theorem (Wadsworth). Let $M \in \Lambda$. Then

$$M \text{ is solvable} \iff M \text{ has a hnf}$$

Böhm-trees

Let $M \in \Lambda$. The Böhm-tree of M, notation $\mathrm{BT}(M)$, is defined by

$$\mathrm{BT}(M) \quad = \quad \uparrow$$

if M has no hnf

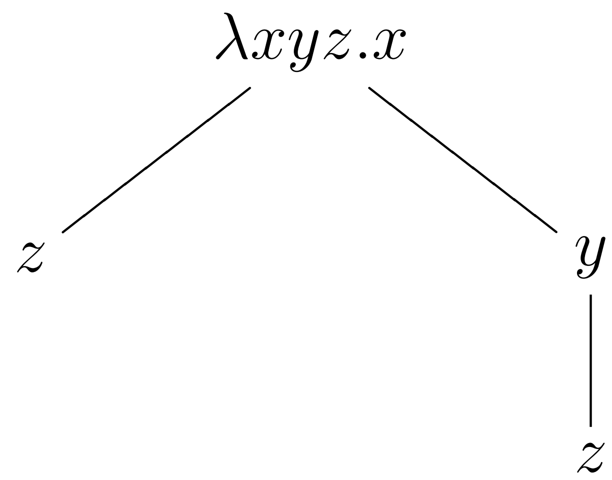
$$= \quad \begin{array}{ccc} & \lambda\vec{x}.y & \\ \swarrow & & \searrow \\ \mathrm{BT}(N_1) & & \mathrm{BT}(N_n) \end{array}$$

if $\lambda\vec{x}.yN_1 \ldots N_n$ is the phnf of M

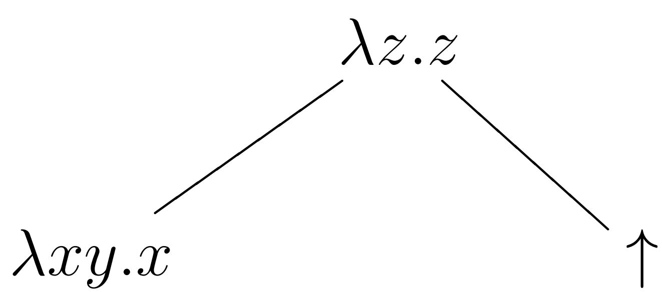
Examples of Böhm-trees

$$\mathrm{BT}(\Delta\Delta) = \uparrow$$

$$\mathrm{BT}(\mathsf{S}) =$$



$$\mathrm{BT}(\langle \mathsf{K}, \Delta\Delta \rangle) =$$



We want to compute $\mathrm{BT}(\mathsf{Y})$

We know $\mathsf{Y} \triangleq \lambda f.\omega_f\omega_f$ with $\omega_f \triangleq \lambda x.f(xx)$

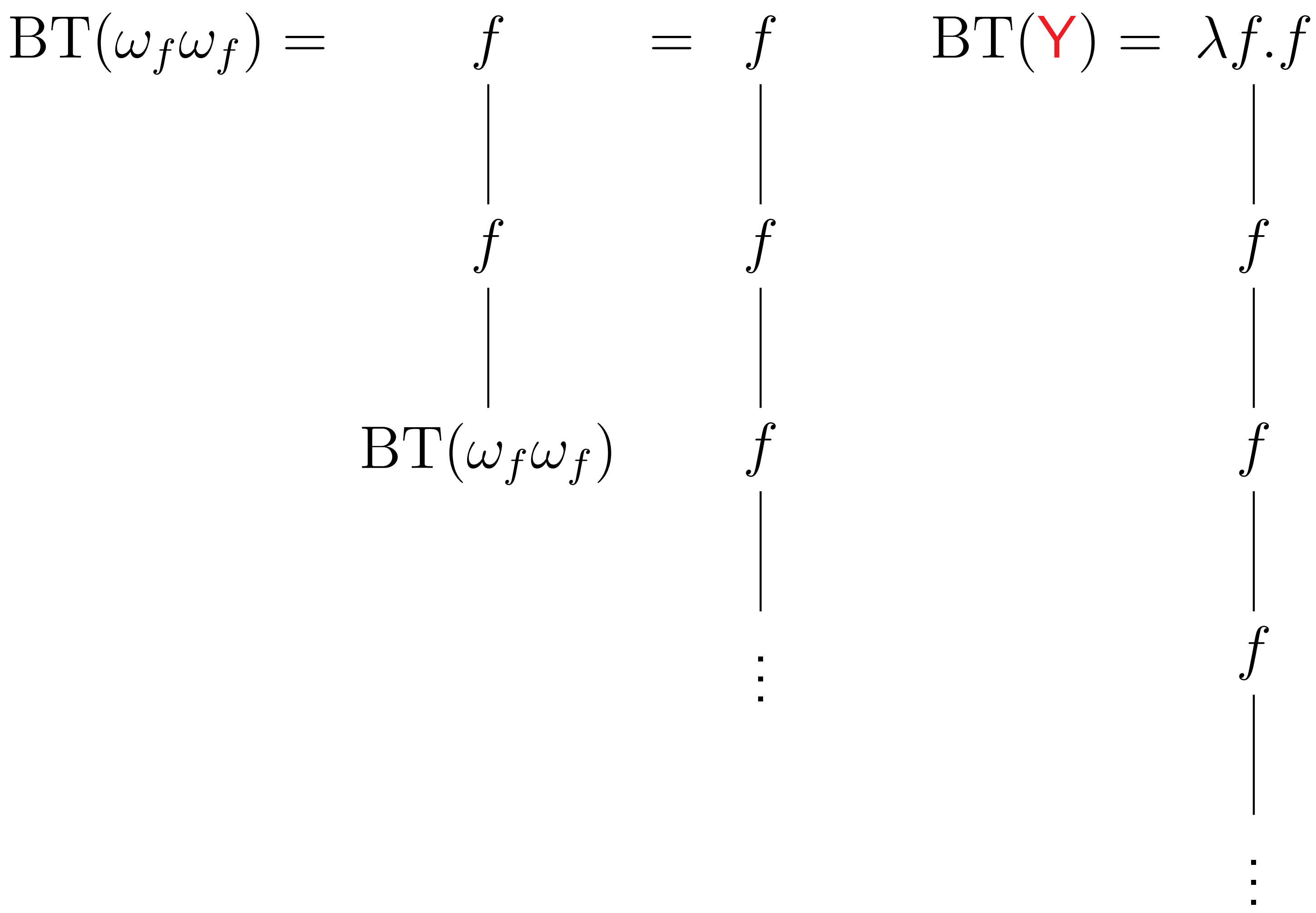
Hence the phnf of Y is $\lambda f.f(\omega_f\omega_f)$

$$\mathrm{BT}(\mathsf{Y}) \quad = \quad \begin{array}{c} \lambda f.f \\ | \\ \mathrm{BT}(\omega_f\omega_f) \end{array}$$

and

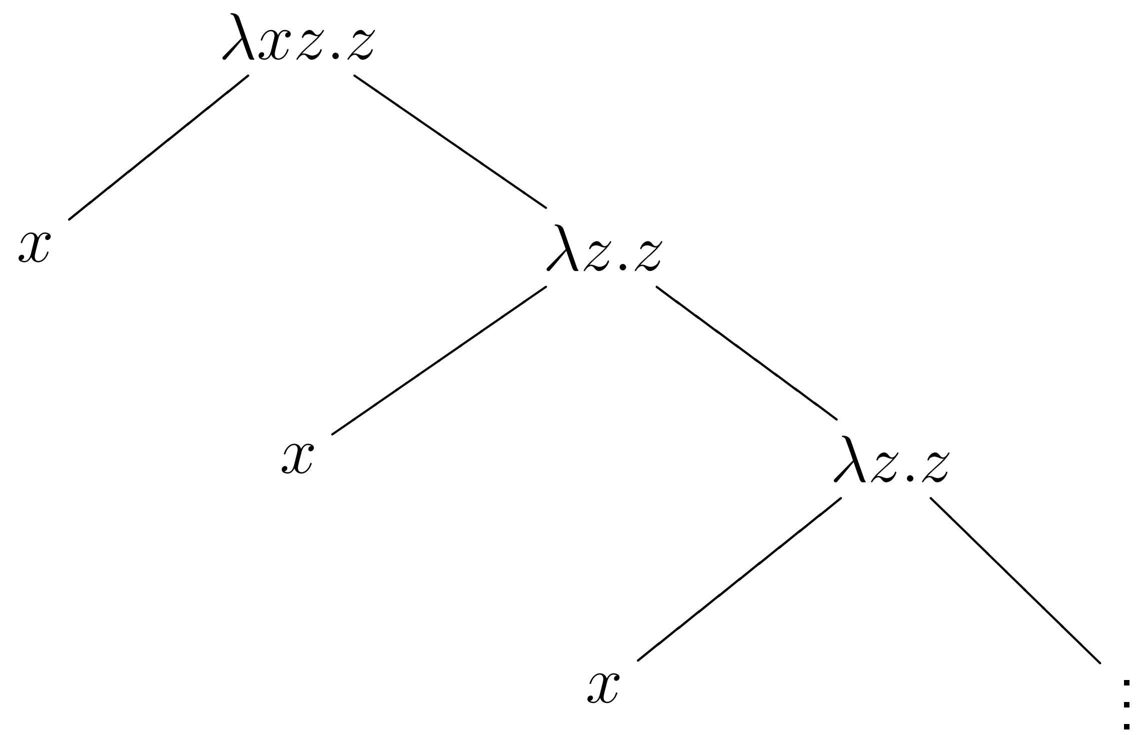
$$\mathrm{BT}(\omega_f\omega_f) \quad = \quad \begin{array}{c} f \\ | \\ \mathrm{BT}(\omega_f\omega_f) \end{array} \quad = \quad \begin{array}{c} f \\ | \\ f \\ | \\ \mathrm{BT}(\omega_f\omega_f) \end{array}$$

BT(Y)



Constructing terms with a given BT

We like a term M with $\mathrm{BT}(M)$ as follows



Take $M \triangleq \lambda x.[x, x, x, \ldots] \triangleq \lambda x.Nx$, with

$Nx = \langle x, Nx \rangle \triangleq \lambda z.zx(Nx)$. So we can take

$N \triangleq \mathsf{Y}(\lambda nxz.zx(nx))$

To be continued!