Lambda Calculus

Week 2

Representing computable functions

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Def. (i) A term $M \in \Lambda$ is called *closed* if $\mathrm{FV}(M) = \emptyset$

(ii) $\Lambda^{\emptyset} = \{M \in \Lambda \mid \mathrm{FV}(M) = \emptyset\}$

Def. Let $M \in \Lambda$. Then

(i) M is a *(β-)normal form* (nf) if for no $N \in \Lambda$ one has $M \to_\beta N$

(ii) M has a nf N if $M =_\beta N$ and N is a nf

Prop. (i) If M is a nf and $M \twoheadrightarrow_\beta N$, then $M \equiv N$

(ii) If M has nf N, then $M \twoheadrightarrow_\beta N$

(iii) A term has at most one nf

Proof. (i) If $M \twoheadrightarrow_\beta N$, then this goes in $k \geq 0$ steps. Then $k = 0$, as M is a nf.

(ii) If $M =_\beta N$, then $M \twoheadrightarrow_\beta Z_\beta \twoheadleftarrow N$ by CR, hence $Z \equiv N$ by (i), as N is a nf.

(iii) If $M =_\beta N_1 =_\beta N_2$, then $N_1 =_\beta N_2$, hence by (ii) $N_1 \twoheadrightarrow_\beta N_2$, as N_2 is a nf, so $N_1 \equiv N_2$ by (i), as N_1 is a nf. ■

A *numerical function* is a partial map $\psi\colon \mathbb{N}^k \to \mathbb{N}$, $k \geq 0$

Such a ψ is *defined at $\vec{n}$*, notation $\psi(\vec{n})\downarrow$, if $\psi(\vec{n}) = m$, for some m

otherwise ψ is *undefined at $\vec{n}$*, notation $\psi(\vec{n})\uparrow$

A *computable function* is a partial computable function that is total

The *initial (computable) functions* are

$$
\begin{array}{rcll}
0 & : & \mathbb{N} & \\
S^+ & : & \mathbb{N} \to \mathbb{N} & \\
U_i^k & : & \mathbb{N}^k \to \mathbb{N} & \quad \text{defined by } U_i^k(n_1, \ldots, n_k) = n_i
\end{array}
$$

Let $\mathcal{A}$ be a class of numeric functions

(i) $\mathcal{A}$ is *closed under composition* if $\chi, \psi_1, \ldots, \psi_m \in \mathcal{A} \Rightarrow \varphi \in \mathcal{A}$, with φ defined by

$$\varphi(\vec{n}) = \chi(\psi_1(\vec{n}), \ldots, \psi_m(\vec{n}))$$

(ii) $\mathcal{A}$ is *closed under primitive recursion* if $\chi, \psi \in \mathcal{A} \Rightarrow \varphi \in \mathcal{A}$, with φ defined by

$$\begin{array}{rclrcl} \varphi(0, \vec{n}) & = & \chi(\vec{n}), & \varphi(0) & = & n_0 \\ \varphi(k+1, \vec{n}) & = & \psi(\varphi(k, \vec{n}), k, \vec{n}) & \varphi(k+1) & = & \psi(\varphi(k), k) \end{array}$$

(iii) $\mathcal{A}$ is *closed under minimalization* if $\chi \in \mathcal{A} \Rightarrow \varphi \in \mathcal{A}$, with φ defined by

$$\varphi(\vec{n}) = \mu m[\chi(\vec{n}, m) = 0]$$

The *partial computable functions* form the least class

- containing the initial functions
- closed under composition, primitive recursion and minimalization

A *total function* is a partial function that is always defined

Church's numerals

$$\begin{aligned}
\mathsf{c}_0 &= \lambda fx.x &&= (\lambda f(\lambda xx)) \\
\mathsf{c}_1 &= \lambda fx.fx &&= (\lambda f(\lambda x(fx))) \\
\mathsf{c}_2 &= \lambda fx.f(fx) &&= (\lambda f(\lambda x(f(fx)))) \\
& \dots \\
\mathsf{c}_n &= \lambda fx.f^{(n)}(x)
\end{aligned}$$

There are terms plus, times satisfying

$$\begin{aligned}
\text{plus } \mathsf{c}_n \ \mathsf{c}_m &=_\beta \mathsf{c}_{n+m} \\
\text{times } \mathsf{c}_n \ \mathsf{c}_m &=_\beta \mathsf{c}_{n \cdot m}
\end{aligned}$$

Take

$$\begin{aligned}
\text{plus} &\equiv \lambda nmfx.nf(mfx) \\
\text{times} &\equiv \lambda nmfx.m(\lambda y.nfy)x
\end{aligned}$$

Then

$$\text{plus } \mathsf{c}_n \ \mathsf{c}_m = \lambda fx.\mathsf{c}_n f(\mathsf{c}_m fx) = \lambda fx.f^n(f^m x) = \lambda fx.f^{n+m}x$$

Def. A function $\psi : \mathbb{N}^k \to \mathbb{N}$ is *λ-definable* if for some $F \in \Lambda^{\emptyset}$ one has

$$\begin{array}{rcll} \psi(\vec{n}) = m & \Rightarrow & F\mathsf{c}_{n_1} \ldots \mathsf{c}_{n_k} =_\beta \mathsf{c}_m & \\ \psi(\vec{n}) = \uparrow & \Rightarrow & F\mathsf{c}_{n_1} \ldots \mathsf{c}_{n_k} & \text{has no nf} \end{array}$$

We say that F λ-defines ψ

Then also

$$\begin{array}{rcll} \psi(\vec{n}) = m & \Leftrightarrow & F\mathsf{c}_{n_1} \ldots \mathsf{c}_{n_k} =_\beta \mathsf{c}_m & \\ \psi(\vec{n}) = \uparrow & \Leftrightarrow & F\mathsf{c}_{n_1} \ldots \mathsf{c}_{n_k} & \text{has no nf} \end{array}$$

and

$$\psi(\vec{n})\downarrow \;\Rightarrow\; F\mathsf{c}_{n_1} \ldots \mathsf{c}_{n_k} =_\beta \mathsf{c}_{\psi(\vec{n})}$$

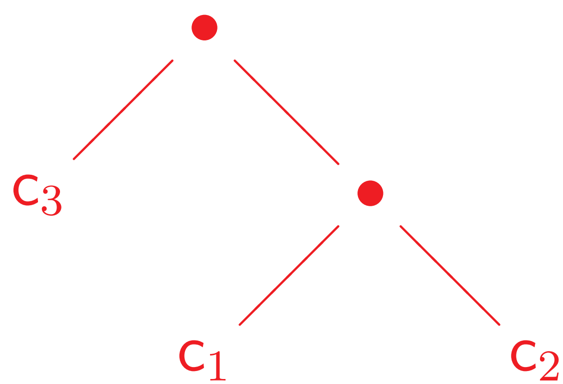
Booleans $\mathsf{true} \triangleq \lambda xy.x$, $\mathsf{false} \triangleq \lambda xy.y$

Then $\mathsf{if}\ b\ \mathsf{then}\ M\ \mathsf{else}\ N \triangleq bMN$

Pairs $\langle M, N\rangle \triangleq \lambda f.fMN$. Then

$$\langle M, N\rangle\,\mathsf{true} = M \quad \langle M, N\rangle\,\mathsf{false} = N$$

Trees



becomes $\lambda p.p\,\mathsf{c}_3(p\,\mathsf{c}_1\,\mathsf{c}_2)$

Mirroring trees is λ-defined by $\mathtt{mirror} = \lambda t\lambda p.t(\lambda ab.pba)$:

$$\mathtt{mirror}(\lambda p.p\,\mathsf{c}_3(p\,\mathsf{c}_1\,\mathsf{c}_2)) = \lambda p.p\,(p\,\mathsf{c}_2\,\mathsf{c}_1)\,\mathsf{c}_3$$

Thm. Let f be a total numeric function. Then

$$f \text{ is computable} \;\Leftrightarrow\; f \text{ is } \lambda\text{-definable}$$

The assumption 'total' can be dropped

Proof. ($\Rightarrow$) (By 'induction') The initial functions can be λ-defined by the terms

$$\mathsf{c}_0, \quad \mathsf{S}^+ \equiv \lambda nfx.f(nfx), \quad \mathsf{U}_\mathsf{i}^\mathsf{k} \equiv \lambda x_1 \ldots x_k.x_i$$

Suppose $g, h_1, \ldots, h_n$ are λ-definable by G, H. Then their composition by

$$F \triangleq \lambda \vec{x}.G(H_1\vec{x}) \ldots (H_n\vec{x})$$

Suppose g is λ-defined by G. Define $f(0) = 7$, $f(n+1) = g(f(n), n)$.

In order to λ-define f we first do $\overline{f}(n) = \langle n, f(n) \rangle$.

Note that $\overline{f}(0) = \langle 0, 7 \rangle$ and $\overline{f}(n+1) = t(\overline{f}(n))$, where $t\langle x, y \rangle = \langle x+1, g(y,x) \rangle$
Then f is λ-defined by

$$F \triangleq \lambda n.nT\langle \mathsf{c}_0, \mathsf{c}_7 \rangle\, \mathsf{false},$$

with $T \triangleq \lambda p.\langle \mathsf{S}^+(p\,\mathsf{true}), G(p\,\mathsf{false})(p\,\mathsf{true})\rangle$. Verify that

$T\langle \mathsf{c}_k, \mathsf{c}_{f(k)} \rangle = \langle \mathsf{c}_{k+1}, \mathsf{c}_{f(k+1)} \rangle$, hence by induction $T^n\langle \mathsf{c}_0, \mathsf{c}_7 \rangle = \langle \mathsf{c}_n, \mathsf{c}_{f(n)} \rangle$, so

$F\mathsf{c}_n = T^n\langle \mathsf{c}_0, \mathsf{c}_7 \rangle\, \mathsf{false} = \langle \mathsf{c}_n, \mathsf{c}_{f(n)}\, \mathsf{false} \rangle = \mathsf{c}_{f(n)}$

Let g be defined by G. Define $f(n) = \mu m.[g(n, m) = 0]$.

Then f is defined by $F = \lambda x.H_F x \mathsf{c}_0$, with

$$H_F x n = \text{ if } (Gn = \mathsf{c}_0) \text{ then } n \text{ else } H_F x(\mathsf{S}^+ n),$$

where "$Gn = \mathsf{c}_0$" is an abbreviation for $Gn(\mathsf{K}\,\mathsf{false})\,\mathsf{true}$

Thus we have seen that the class of λ-definable functions contains the initial functions and is closed under substitution, primitive recursion and minimalization. Hence it contains all computable functions.

($\Leftarrow$) Suppose that F defines f. Then

$$f(n) = m \;\Leftrightarrow\; \boldsymbol{\lambda} \vdash F\mathsf{c}_n = \mathsf{c}_m$$

is an enumerable relation (since the axioms of $\boldsymbol{\lambda}$ are decidable).

Hence by computability theory f is computable. $\square$

$$\begin{array}{rcl} \text{computations} & \rightsquigarrow & \text{termination} \\ \text{processes} & \rightsquigarrow & \text{continuation} \end{array}$$

Simplest continuation

Let $\Delta = \lambda x.xx$. Then

$$\begin{array}{rcl} \Delta\,\Delta & = & (\lambda x.xx)\,\Delta \\ & = & \Delta\,\Delta \end{array}$$

This can be done in interesting ways

Given $C[\vec{x}, f] = \ldots \vec{x} \ldots f \ldots$, there is a term F such that

$$F\vec{x} = C[\vec{x}, F]$$

Prop. Given $F_1, \ldots, F_n \in \Lambda$. Then there exists $A_1, \ldots, A_n \in \Lambda$ such that

$$\begin{array}{rcl} A_1 & \twoheadrightarrow_\beta & F_1 \vec{A} \\ & \ldots & \\ A_n & \twoheadrightarrow_\beta & F_n \vec{A} \end{array}$$

Proof. For $X_1, \ldots, X_n \in \Lambda$ write $\langle X_1, \ldots, X_n \rangle \triangleq \lambda z.zX_1 \ldots X_n$. Then

$$\langle X_1, \ldots, X_n \rangle \mathsf{U}_i^n \twoheadrightarrow_\beta X_i.$$

Let

$$X \twoheadrightarrow_\beta \langle F_1(X\mathsf{U}_1^n) \ldots (X\mathsf{U}_n^n), \ldots, F_n(X\mathsf{U}_1^n) \ldots (X\mathsf{U}_n^n) \rangle$$

Hence taking $A_i \triangleq X\mathsf{U}_i^n$ we have

$$A_i \triangleq X\mathsf{U}_i^n \twoheadrightarrow_\beta F_i(X\mathsf{U}_1^n) \ldots (X\mathsf{U}_n^n) \triangleq F_i A_1 \ldots A_n. \;\blacksquare$$