Comments on Infinite Indexed Sets in Computer Algebra Systems

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1 Introduction

Computing with "sets" is well-explored in the programming-language and data-structure literature. Many languages have one or more set representations as well as operations for these sets. Unfortunately, the notion of set in mathematics is far more elaborate than the supported notion. The programming languages' notations and operations work for explicit finite sets only, not (for example), the set of all odd primes.

Representing and manipulating *all* kinds of sets and set descriptions, including infinite set notations, or cases where the set is described algebraically, geometrically, logically, functionally or using some other "implicit" method is an open-ended challenge.

In this paper we address a subset of the problem prompted by an application of a computer algebra system (CAS). We show how we can "automatically" provide some simplification of set descriptions and other operations. Our motivation comes from problems in which we obtained infinite sets such as $\{n\pi\}$ for integer n, and must compute with these further. These sets can be easily generated by a CAS asked to solve $\sin x = 0$ for x. To date, simplification of operations on such resulting sets are generally unavailable from CAS, making the result suitable for immediate human consumption but almost useless in building other tools.

2 Set notation

There are many common notations for designating sets. Large sections of various font alphabets are consumed by special conventions for naming sets in algebra and analysis. We do not intend to close off future extension or further discussions, but to keep our discussion finite, we must make some choices to simplify our problem. We hope to do so while retaining some useful results.

2.1 Basic named sets

The set of natural numbers $\{0, 1, 2, 3, \cdots\}$, will be denoted $\mathbb{N}$. We use it in the *index set* designation of other sets.

An alternative index set, of all integers, $\{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$, will be denoted $\mathbb{Z}$. We will generally use identifiers n, m, k to mean variables restricted over $\mathbb{Z}$. Thus $n > -1$ means non-negative integers. (that is, $n > -1$ also means the same as $n \in \mathbb{N}$). Mathematica allows the following built-in concepts to be used for basic sets: `Algebraics, Booleans, Complexes, Integers, Primes, Rationals, Reals.`

2.2 Finite Lists

While we focus on infinite sets, we must also allow for the "easy" cases.

- The empty set $\{\}$, sometimes written $\emptyset$.

- Singletons: sets with only a single value. Arguably all results for CAS commands should be potentially *sets* of answers. The familar case is the singleton set with a single value. For example $1+1 \rightarrow 2$ might be $1+1 \rightarrow \{2\}$. This allows for consistent treatment of $\sqrt{9} \rightarrow \{3,-3\}$ or $\{1^{1/3} \rightarrow \{1,\ -1/2+1/2\,i\sqrt{3},\ -1/2-1/2\,i\sqrt{3}\}$

  An astute observer might say that if $\sqrt{9}$ has two values, so does $\sqrt{3}$ in that last expression. So are there really five values, give the bifurcation of the $\sqrt{3}$ values? What can be meant by that triple of expressions? In fact, we can choose either of the two possible particular valuea of $\sqrt{3}$and, if used consistently, obtain the same set of roots (although permuted). If that set of roots is separated and the three values interpreted in different contexts (and with inconsistent signs for $\sqrt{3}$) we may not be so safe!

- Other explicit lists: all elements are provided: E.g. {a,b,c}.

- Multisets: sets including possible multiple occurrences of an element. E.g. $\{a,b,a,c\}$. (The kinds of applications of multisets that come to mind are not very compelling. Consider the set of zeros of a polynomial. It is more likely that representing them as a multiset would not be as useful as a representation for a set of pairs (*value, multiplicity*), or two lists, of roots and multiplicities. Either would be easier for a program to handle.) We will generally try to ignore extra duplicated elements and will try not to distinguish $\{a,b,a,c\}$ from $\{a,b,c\}$.

2.3 Plus-minus Expressions

This notation (as input or output) is not commonly accepted in computer algebra systems, but is widely used in applied mathematics texts.

$-b/2 \pm \sqrt{d}$ is a set of two expressions.

$a \pm b \pm c$ is a set of 4 expressions, though $a \pm b \mp c$ is a set of 2 expressions.

2.4 Indexed Sets

The common notation is $\{f(n) \mid n \in \mathbb{N}\}$

A better notation would make clear the relationship between the two several occurrences of n. In particular, only when n occurs "free" on the left, it should be bound to the n on the right. (A non-free, i.e. bound n occurs in the integrand "n^2dn" here: $\{n+\int_0^n n^2dn \mid n \in \mathbb{N}\}$) and only the "external" n ranges over $\mathbb{N}$.

A clearer notation which uses explicit ideas is available from the lambda calculus, which is simple and familiar to anyone who has learned Lisp or logic: We will use $\{\lambda(n)f(n) \mid \mathbb{N}\}$ for emphasis or when it seems appropriate to be more formal.

2.5 Compound Sets

Given any of the representations for sets A, B, C, it may be infeasible to simplify expressions like $A \cap B$ or $A \cup B$ or the complement $\bar{A}$. Our representation is too general to allow explicit computation of unions or intersections in all cases. The typical explicit sets provided in "set" packages are different in this respect, since their sets are finite (often very finite, e.g. when set elements are bits in a finite string).

Regardless of the underlying representation, programs doing set simplification can potentially use various identity laws, distribution, deMorgan's laws, rather than explicit iteration over the elements. While useful around the margins and dealing with left-over pieces of sets, we suspect that general rules of this nature are not going to be very effective for our applications.

3 Operations

Some of our set operations may not be algorithmically computable *in general*. In most cases this does not prevent us from writing a program to perform these operations *sometimes*, and in other cases leave the results unresolved. When a set expression is unresolved, if is carried along symbolically, rather than in some enumerative structure. Consider $R = |x|x \notin x$, the set of all sets that are not members of themselves, suggested by Bertrand Russell in 1901, and leading to "Russell's Paradox." That this set cannot be constructed is certainly a problem, but not immediately as pressing as problems with (in principle) constructible sets which appear to require mechanisms we have not yet devised.

- Test for empty set.
- Choose an element. (and remove it from a set.) Generate the next element, etc. (This may impose an order on the set. This may be done in an indexed set by choosing the element with the least index, if that is possible.) A (generally not computable) alternative is to extract smallest element.
- Membership. Is $x \in S$? e.g. is $0 \in \{\lambda(n)\ 2 \times n\ \ | \mathbb{N}\}$?
- Intersection: Find an expression simpler than $A \cap B$ for an intersection. This is not possible in general, but it is helpful to be able to compute, for example that $\{\lambda(n)(2 \times n + 1)\ \ | \mathbb{N}\}\ \cap\ \{\lambda(n)2 \times n\ \ | \mathbb{N}\}$ can be expressed more simply, namely as the empty set $\emptyset$.
- Union: Find a simple expression than $A \cup B$ for a union. For $\{\lambda(n)2 \times (n+1)\ \ | \mathbb{N}\}\ \cup\ \{\lambda(n)2 \times n\ \ | \mathbb{N}\}$ we can use the set $\mathbb{N}$.
- Cardinality: the number of elements, which may be infinite. Unique cardinality (counting repeated elements once). For some descriptions the cardinality may not be computable, or we may not be clever enough to compute it. Sets defined by pipes typically fall in this category.
- Cross product: $A \times B$, the set of ordered pairs from sets A and B.
- Other operations may be added to this list.

4 Techniques

- Reduction to canonical index names. e.g. operations on sets should *not* be able to distinguish $\{\lambda(n)(2 \times n + 1)\ \ | \mathbb{N}\}$ from $\{\lambda(m)(2 \times m + 1)\ \ | \mathbb{N}\}$, each of which is the set of non-negative odd integers.
- Black box generators. In the sense of programming language pipes or streams, consider a set representation in which you can examine or extract the next one of its elements. In practice two values are returned: the extracted element and a representation of the set of *remaining* elements. Thus one might have a program representing the non-negative integers. Extracting the first element 0 and a new set, the set of integers beginning at 1. In the lisp dialect of Scheme, [1], the expression of this set may be as simple as this:

```
(define (integers-from n)(make-pipe n (integers-from (1+ n 1))))
(define non-neg-integers (integers-from 0))
```

From this can be created, using other programs working as filters or mapping functions, a wide variety of computational "pipes". For example, pipes that contain only primes are one of the sample illustrations in a popular text [1].

- Compare to well-known sets: $\mathbb{Z}$, $\mathbb{N}$. $\mathbb{Q}$, (rationals) $\emptyset = \{\}$ to see if a relationship (subset, equality?) holds.
- Simplifications include all operations (distributivity, deMorgan's laws) defined for sets generally in the same CAS.

5 Requirements, benchmarks

We would like to be able to consider solving the problems in this (growing) list.

1. $\{2n \mid n \in \mathbb{N}\} = \{2 * m \mid m \in \mathbb{N}\} = \{2 * k + 4 \mid k > -3\}$

   To notice this equality, the usual approach in a CAS is to find a *canonical form* to which each of these can be simplified. (See lambda-binding above).

2. $\{2n + 1 \mid n \in \mathbb{N}\} \cup \{2 * n \mid n \in \mathbb{N}\} = \mathbb{N}$

   Even and odd numbers are all there can be.

3. $\{2n + 1 \mid n \in \mathbb{N}\} \cap \{2n \mid n \in \mathbb{N}\} = \{\}$

   Even and odd numbers don't overlap.

4. $\{n - 1 \mid n \geq 0\} \cap \{m \mid m \leq 0\} = \{-1,\ 0\}$

   Finite sets can result from intersections of infinite sets.

5. $\{2n \mid n \in \mathbb{N}\} \cup \{3m \mid m \in \mathbb{N}\} = \{n \mid ((n \bmod 3 = 0) \vee (n \bmod 2 = 0)) \ \wedge \ n \in \mathbb{N}\}$. This right-hand expression is probably not more attractive than the left-side expression. Arbitrary manipulation of the index set is clearly going to lead to problems.

   > Let $P = \{n \mid \text{prime}(n)\}$ and $R = \{(p + q) | (p \in P) \wedge (q \in P)\}$ If we can prove the set of even numbers greater than 4 is a subset of R we have proved Goldbach's conjecture[1].

6. $\{2n \mid n \in \mathbb{N}\} \cap \{3m \mid m >\in \mathbb{N}\} = \{n \mid n \bmod 6 = 0) \mid n \in \mathbb{N}\}$

   How can we deal with this?

   $\{n^2 \mid n \in \mathbb{N}\} = \{m^2 - 2m + 1 \mid m \in \mathbb{N}\}$ Each includes all integers that are squared.

6 Applications

6.1 Symmetry, periodicity in integration

Our initial motivation for beginning this exploration was an application that requires detecting periodicity of a function. In this application, in the process of computing a definite integral we first check for symmetry or antisymmetry of an integrand [2] around some point (to be found). Our method depends on solving a related shifted equation whose solution can be an infinite set of the nature required above. Continuing the computation in a CAS unattended, that is, with the human taken *out of the loop*, requires further operations to be done sight unseen. In particular, the intersection of two sets, each an infinite set of points, might be finite (even empty) or infinite. An algorithmic approach is required. Similar questions arise about set unions.

[1] Namely all positive even integers $n \geq 4$ can be expressed as the sum of two primes. A reward of $1,000,000 for such a proof was offered by a publisher (Faber and Faber), although it is now expired.

6.2 Other related problems

The notational and simplification issues here have parallels in dealing with other mathematical objects. For example, consider $\sum_{n=1}^{\infty} f(n) + \sum_{m=1}^{\infty} g(m)$. Simplification of this expression requires renaming of one (or both) of the index sets; the same lambda-binding idea holds. It requires aligning the index sets in $\sum_{n=1}^{\infty} f(n)x^n + \sum_{m=0}^{\infty} g(m)x^m$.

7 Plan of attack

As stated initially, we must make some judicious choices and fix some of the many variabilities.

We are not interested in discussing simplification of (other kinds of) sets that may already be included in a CAS. Thus we assume $A \cup A = A \cap A \rightarrow A$ and similar rules are known and applied when possible. Since our motivation was to be able to compute further with cases as returned from CAS solve programs, let us look at them again.

Consider the set we got from solving $\sin x = 0$ for x. The expression $\arcsin(0)$ is actually a set, $n\pi$ meaning $\{n\pi \mid n \in \mathbb{N}\}$[2]

Here is the kind of computations might we wish to do with this set:

What solutions does the equation $\sin(x) = 0$ have in common with $\sin(x/2) = 0$? The two solution sets are $\{n\pi \mid n \in \mathbb{N}\}$ and $\{2m\pi \mid m \in \mathbb{N}\}$. Their intersection is found by (a) assuring that the index range is identical, and (b) solving $n\pi = 2m\pi$ for (say) n, resulting in $n = 2m$. The intersection result is thus $\{2m\pi \mid m \in \mathbb{N}\}$. It is not appropriate to solve for m to get $m = n/2$ because m must be in $\mathbb{N}$, and half-integer values are not permitted. We must compute this explicitly, as shown below.

7.1 What is the union of the two sets?

This can be rather complicated since we are searching for a simplified form. Here is a heuristic that sometimes helps, given that A and B are simplified set notations. If $A \subseteq B$ then $A \cup B = B$. Constructively can we show $x \in A \;\Rightarrow\; x \in B$? In order to show that for each value $2m\pi$ there is a value $n\pi$, set them equal and solve for n. We must conclude $(n = 2m \;\wedge\; m \in \mathbb{N}) \Rightarrow n \in \mathbb{N}$ by noting that multiplication by 2 maps $\mathbb{N}$ into $\mathbb{N}$.

7.2 When are two sets equal?

Consider a problem that will turn out to be too hard. We encountered the question `solve(sin(x+c)=sin(x),c);` Macsyma says:

$$\left\{c = 2\,\arctan\left(\frac{\cos x}{\sin x}\right) + 2\,\pi\,\mathrm{n}_1, c = 2\,\pi\,\mathrm{n}_2\right\}.$$

This is kind of hard to fathom, so substituting $y - c/2$ for x and then expanding the sines of sums, we get:

$$2\,\sin\left(c/2\right)\,\cos y \;=\; 0$$

The first factor leads to solutions of the form $c = 2\arcsin 0$. The second factor leads to $x + c/2 = \arccos 0$ or $c = 2 \cdot (\arccos 0 - x)$. Combining these we can claim the solution is

$$\{n\pi\} \cup \{(2m+1)\pi - 2x\}.$$

[2] A possible quibble: if $n\pi$ is an angle, then can we say that—as angles—$0 = 2\pi$ in which case the set of unique solutions is the set of only two: $\{0,\ \pi\}$. We reject this because there are other ways of looking at arcsin where the angle continues to evolve indefinitely to higher numbers.

Here is another approach to the same question. Let us convert to exponential form first, then use the `radcan` simplification command. In this case Macsyma gives:

$$\{-2\,x + 2\,\pi\,n + \pi\} \cup \{0\}.$$

Actually, that $c = 0$ solution is defective; it is claiming that $e^{ic} = 1$ has the (sole) solution $c = 0$ instead of $c = 2k\pi$. A corrected solution would then be

$$\{-2\,x + 2\,\pi\,n + \pi\} \cup \{\ 2k\pi\}.$$

Mathematica says

$$\{-x + \arcsin(\sin(x))\}.$$

This expression is a piecewise continuous function of x that is periodically constant.

- When $-\pi/2 \le x \le \pi/2$ then $c = 0$.
- When $\pi/2 \le x \le 3\pi/2$ then $c = -2x + \pi$.
- When $3\pi/2 \le x \le 5\pi/2$ then $c = -2\pi$. Etc.

Maple finds only the solution is $c = 0$. True, but not very complete.

Comparing the above sets for all $x \in D$ (D might be real or complex numbers) is complicated, and not something we have programmed. In fact, the Mathematica set does not include solutions like $c = -2x + 5\pi$ which are provided in the Macsyma solution, so the sets are *not* equal.

The sets differ on a grosser level. The Macsyma solutions comprise an infinite set of lines with slope -2 spaced 2π apart. The Mathematica solution is single-valued (one member for each real x).

8 Prospects for implementation

8.1 Representation of sets

A basic set is a pair: The first element is $\lambda(n)f(n)$. e.g. $\lambda(n)n^2$ for set of squares. The second element is the domain for n, with default $n \in \mathbb{N}$. Other possibilities include $n > a$, $n < a$. The expression a must be integer-valued, although not necessarily an explicit integer. As will be evident shortly, we do not expect to guarantee uniqueness of representation expect in trivial cases, and furthermore there may be sets that cannot be represented (conveniently) this way, but must be expressed by combinations of these sets.

8.2 Explicit Operations

8.2.1 change of index, upward

```
(lambda(n)f(n); n>a)
 if a is -1,  change to (lambda(n)f(n); n in N)
 if a is not -1, change  to  ((lambda(n)(lambda(q) f(q))(n+a)) n in N)
  ; q is not free in f.
```

8.2.2 change of index, downward

```
 (lambda(n)f(n); n<a)
 if a is 1,  change to (lambda(n)f(-n); n in N)
 if a is not 1,  change to ((lambda(n)(lambda(q) f(q))(-n+a)) n in N)
; q is not free in f.
```

```
(lambda(n)(lambda(q) f(q))(n+q)) can ordinarily be rewritten as (lambda(n)(f(n+q))).
 [assuming purely functional expression f.]
```

8.2.3 Element test

```
   e is in ( lambda(n)f(n), n in N) if there is a solution n=k to
  f(n)=e and k is in N. Requires commands solve, and natnump.
Natnump(x):= if x is explicit natural number
             or x is declared to be a natural number
             or (x= y+z and natnump(y) and natnump(z))
             or (x= y*z and natnump(y) and natnump(z))
             or (x= y-z and natnump(y) and natnump(z))
             or (x= y^z and natnump(y) and natnump(z))

  may fail to identify some natnums.

fails on (n+1)*n/2 which is a natural number. either n or n+1
is even and thus the 2 always divides one or the other

Heuristic:  for each indeterminate {v1,v2,..) substitute
a random integer. If the result is  an integer, suggest
that the expression  might be natnump. Test repeatedly to be more
confident.
```

8.2.4 Subset test

```
  is  ( lambda(n)f(n), n in N)  a subset of
      ( lambda(m)g(m), m in N) ?

  solve the equation f(n)=g(m) for n=k
  test natnump(k) assuming m in N?
```

8.2.5 Equality test

```
Equality could be defined by $$(A \subset B) \wedge (B \subset A).$$
The application of this definition is probably not directly useful.
```

8.2.6 Intersection

```
   Let A=  ( lambda(n)f(n), n in N)
       B= ( lambda(m)g(m), m in N).
```

$A \cap B$ is the set of all e such that there is an $n \in N$ and an $m \in N$ such that $e = f(n) = g(m)$.

Example, $A = \{n^2\}$, $B = \{2m\}$. $0, 1, 4, 9, 16, 25, ... \cap 0, 4, 8, 16, .. = \{0, 4, 16, ...\}$

The common elements can be found by setting $n^2 = 2m$ and solving the polynomial diophantine equation for $n,\ m \in N$.

Since we cannot expect to solve arbitrary polynomial equations, we can consider sub-classes of the problem such as restricting the functions f, g, to linear functions of n

Examples: Consider $\{2n+1\} \cap \{2m\}$ or $\{2n+5\} \cap \{2m\}$.

For the first problem x in the intersection, $x = 2n+1$ for some $n \in \mathbb{N}$ and also $x = 2m$ for some $m \in \mathbb{N}$.

Thus $2n+1 = 2m$, and we can try `solve(2*n+1=2*m,m)` which gives $m = (2n+1)/2$. This cannot be shown to be a natnum by our previous method; substitution of random integers also shows, at least some of the time, it is *not* natnum.

`solve(2*n+5=2*m,m)` gives $m = (2n+5)/2$, same argument.

How about $\{12kn\} \cap \{20km\}$, where `solve` gives $m = 12/20n$? This is apparently not a natnum on its face. Caution is required here: assuming there is no solution *because we can't find it with this weak test* leads to an incorrect conclusion.

How can we provide a more general approach?

n and m, with constants a, b, c, d for $an + b = cm + d$, consider writing this as $an = cm + (d - b)$ and then consider it as equation mod c: the task then is to solve is $an \mod c = d - b$

How about $\{12kn + 3\} \cap \{20km + 1\}$, which reduces to The question then is to solve for n: in $12n \mod 20 = 18$.

This is a problem in Diophantine equations. Indeed, Mathematica allows this: `Solve[{12n ==18,Modulus==20},n,Mode`
Unfortunately it gives the wrong answer (in version 4.1 at least).

Maple 7's `isolve` program does much better; it immediately finds there are no solutions. In this particular example, the reductions of multiples of 12 modulo 20 are all in the set {0, 4, 8, 12, 16} which does not include the required 18, so the intersection is empty.

There is a substantial literature on solving Diophantine equations (as well as when they cannot be solved.[4]). It is clear that correct solutions for some classes of Diophantine equations are programmable (as in Maple) and we can hope that such a facility will at least occasionally, solve our immediate problems.

9 Conclusion

Further exploration of the possibilities of set representation is required if we are to portray computer algebra systems as effective tools to solve even rather simple problems in analysis. This paper points to some of the difficulties and suggests some approaches. While this paper does not provide a complete solution, the use of Diophantine solvers is at least part of the armamentum. We hope this paper's approach opens up some avenues for exploration. At least we find this preferable to writing (or reading) a paper which consists of abstracting out or creating a small, uninteresting, ungeneralizable, yet solvable variant of a difficult problem, and then defining and lemma-ing it to death.

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References

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