Chapter 18

Models 14.11.2006:1338

In this Chapter filter models, the main tool of Part III on the intersection types, will be introduced. A *filter* is a collection of types closed under intersection ($\cap$) and expansion ($\leq$). It turns out that there is a natural way to define application on such filters. This depends on the order $\leq$ on types and it will be shown for which of the type theories introduced in Chapter 15 the filters will turn out to be models of the untyped lambda calculus.

In Section 18.2 the filter models will be introduced as an applicative structures. Also it will be shown that the value of an untyped lambda term M in this structure is the collection of types that can be assigned to M. In Section 18.3 the approximation theorem will be shown, i.e. the interpretation of a lambda term is the supremum of those of its approximations.

18.1. Lambda models

Given a lambda structure $\mathcal{D}_{F,G} = \langle \mathtt{D}, F, G\rangle$, i.e. a $\mathcal{D}\in\mathbf{ALG}$ with continuous $F : \mathcal{D}\to\mathcal{D}\to\mathcal{D}$ and $G : [\mathcal{D}\to\mathcal{D}]\to\mathcal{D}$, it is well known how one can interprete (untyped) lambda-terms in it. For lambda structures of the form $\mathcal{D} = \mathcal{F}^{\mathcal{T}}$ this interpretation turns out to have a simple form: the interpretation of a lambda term equals the set (actually a filter) of its possible types (in $\mathbb{T}^{\mathcal{T}}$). This will help us to determine for what $\mathcal{T}$ the corresponding filter structure is a lambda-model. This characterization can also be given for the λI-calculus.

18.1.1. Definition. (i) Let $\mathcal{D}$ be a set and Var the set of variables of the untyped lambda calculus. An *environment in* $\mathcal{D}$ is a total map

$$\rho : \mathrm{Var}\to\mathcal{D}.$$

The set of environments in $\mathcal{D}$ is denoted by $\mathsf{Env}_{\mathcal{D}}$.

(ii) If $\rho\in\mathsf{Env}_{\mathcal{D}}$ and $d\in\mathcal{D}$, then $\rho[x := d]$ is the $\rho'\in\mathsf{Env}_{\mathcal{D}}$ defined by

$$\begin{array}{rcll}\rho'(x) & = & d; & \\ \rho'(y) & = & \rho(y), & \text{if } y \neq x.\end{array}$$

The definition of a syntactic lambda-models was given in Barendregt [1984] (Definition 5.3.1) or Hindley and Longo [1980]. We simply call these λ-models.

We introduce also *applicative structures* (Definition 5.1.1 of Barendregt [1984]) and *quasi* λ*-models*.

18.1.2. DEFINITION. (i) An *applicative structure* is a pair $\langle \mathtt{D}, \cdot \rangle$, where $\mathtt{D}$ is a set and $\cdot : \mathtt{D} \times \mathtt{D} \rightarrow \mathtt{D}$ is a binary operation on $\mathtt{D}$.

(ii) A *quasi* λ*-model* is of the form

$$\mathcal{D} = \langle \mathtt{D}, \cdot, [\![\;]\!]^{\mathtt{D}} \rangle,$$

where $\langle \mathtt{D}, \cdot \rangle$ is an applicative structure and $[\![\;]\!]^{\mathcal{D}} : \Lambda \times \mathsf{Env}_{\mathcal{D}} \rightarrow \mathtt{D}$ satisfies the following.

$$\begin{array}{lrcll}
(1) & [\![x]\!]^{\mathtt{D}}_{\rho} & = & \rho(x) & \\
(2) & [\![MN]\!]^{\mathtt{D}}_{\rho} & = & [\![M]\!]^{\mathtt{D}}_{\rho} \cdot [\![N]\!]^{\mathtt{D}}_{\rho} & \\
(3) & [\![\lambda x.M]\!]^{\mathtt{D}}_{\rho} & = & [\![\lambda y.M[x := y]]\!]^{\mathtt{D}}_{\rho} & (\alpha), \\
 & & & \text{provided } y \notin \mathrm{FV}(M), & \\
(4) & \forall d \in \mathcal{D}.[\![M]\!]^{\mathcal{D}}_{\rho[x:=d]} = [\![N]\!]^{\mathcal{D}}_{\rho[x:=d]} & \Rightarrow & [\![\lambda x.M]\!]^{\mathtt{D}}_{\rho} = [\![\lambda x.N]\!]^{\mathtt{D}}_{\rho} & (\xi) \\
(5) & \rho \restriction \mathrm{FV}(M) = \rho' \restriction \mathrm{FV}(N) & \Rightarrow & [\![M]\!]^{\mathtt{D}}_{\rho} = [\![M]\!]^{\mathcal{D}}_{\rho'}. &
\end{array}$$

(iii) A λ*-model* is a quasi λ-model which safisfies:

$$(6) \quad [\![\lambda x.M]\!]^{\mathtt{D}}_{\rho} \cdot d \;=\; [\![M]\!]^{\mathcal{D}}_{\rho[x:=d]} \quad (\boldsymbol{\beta})$$

(iv) A *(quasi)* $\lambda\mathsf{I}$*-model* is defined similarly but replacing Λ by Λ^{I}, the set of $\lambda\mathsf{I}$-terms that require for each abstraction term $\lambda x.M$ that $x \in \mathrm{FV}(M)$. The corresponding clauses are denoted by $(\alpha\mathsf{I})$, $(\boldsymbol{\beta}\mathsf{I})$ and $(\xi\mathsf{I})$.

We will write simply $[\![\;]\!]_{\rho}$ instead of $[\![\;]\!]^{\mathtt{D}}_{\rho}$ when there is no danger of confusion.

We have the following implications.

$$\begin{array}{ccc}
\mathcal{D}\ \lambda\text{-model} & \Longrightarrow & \mathcal{D}\ \lambda\mathsf{I}\text{-model} \\
\Downarrow & & \Downarrow \\
\mathcal{D}\ \text{quasi}\ \lambda\text{-model} & \Longrightarrow & \mathcal{D}\ \text{quasi}\ \lambda\mathsf{I}\text{-model}
\end{array}$$

18.1.3. DEFINITION. Let $\mathcal{D} = \langle \mathtt{D}, \cdot, [\![\;]\!] \rangle$ be a (quasi) $\lambda(\mathsf{I})$-model.

(i) The statement $M = N$, for M, N untyped lambda terms, is *true in* $\mathcal{D}$, notation $\mathcal{D} \models M = N$ iff

$$\forall \rho \in \mathsf{Env}_{D}.[\![M]\!]_{\rho} = [\![N]\!]_{\rho}.$$

(ii) As usual one defines $\mathcal{D} \models \chi$, where χ is any statement built up using first order predicate logic from equations between untyped lambda terms.

(iii) A $\lambda(\mathsf{I})$-model $\mathcal{D}$ is called *extensional* iff

$$\mathcal{D} \models (\forall x.Mx = Nx) \;\Rightarrow\; M = N.$$

(iv) A $\lambda(\mathsf{I})$-model $\mathcal{D}$ is called an $\boldsymbol{\eta}$*-model* iff

$$\mathcal{D} \models \lambda x.Mx = M \text{ for } x \notin \mathrm{FV}(M) \qquad (\boldsymbol{\eta})$$

18.1.4. DEFINITION. (i) Let $\mathcal{D}_{F,G} = \langle \mathtt{D}, F, G\rangle$ be a lambda structure, Definition 17.4.1(i). Then $\mathcal{D}_{F,G}$ induces a quasi λ-model $\langle \mathtt{D}, \cdot, [\![\ ]\!]^{F,G}\rangle$ as follows.

- First we obtain an applicative structure by setting for $d, e \in \mathtt{D}$

$$d \cdot e = F(d)(e).$$

- Then the map $[\![\ ]\!]^{F,G} : \Lambda \times \mathsf{Env}_{\mathcal{D}} \to \mathtt{D}$ as defined as follows.

$$\begin{array}{rcl} [\![x]\!]^{F,G}_\rho &=& \rho(x);\\ [\![MN]\!]^{F,G}_\rho &=& F([\![M]\!]^{F,G}_\rho)([\![N]\!]^{F,G}_\rho);\\ [\![\lambda x.M]\!]^{F,G}_\rho &=& G(\boldsymbol{\lambda} d \in \mathcal{D}.[\![M]\!]^{F,G}_{\rho[x:=d]}). \end{array}$$

Notice that the function $\boldsymbol{\lambda} d \in \mathcal{D}.[\![M]\!]_{\rho[x:=d]}$ used for $[\![\lambda x.M]\!]_\rho$ is continuous.

(ii) Now let $\mathcal{D}_{F,G} = \langle \mathcal{D}, F, G\rangle$ be a *strict* lambda structure, see Definition 17.4.1(ii). Then $\mathcal{D}_{F,G}$ induces a quasi $\lambda\mathsf{I}$-model as above above, changing the clause for $[\![\lambda x.M]\!]^{F,G}_\rho$ into

$$[\![\lambda x.M]\!]^{F,G}_\rho = G(\boldsymbol{\lambda} d \in \mathcal{D}.\ \text{if } d = \bot_{\mathcal{D}} \text{ then } \bot_{\mathcal{D}} \text{ else } [\![M]\!]^{F,G}_{\rho[x:=d]}).$$

18.1.5. PROPOSITION. (i) *Let $\langle \mathtt{D}, F, G\rangle$ be a lambda structure. Then $\langle \mathtt{D}, \cdot, [\![\ ]\!]^{F,G}\rangle$ is a λ-model.*

(ii) *Let $\langle \mathtt{D}, F, G\rangle$ be a strict lambda structure. Then $\langle \mathtt{D}, \cdot, [\![\ ]\!]^{F,G}\rangle$ is a $\lambda\mathsf{I}$-model.*

PROOF. Easy. ∎

The only requirement that a (strict) lambda structure misses to be a $\lambda(\mathsf{I})$-model is the axiom $(\boldsymbol{\beta}(\mathsf{I}))$.

18.1.6. PROPOSITION. (i) *Let $\langle \mathtt{D}, F, G\rangle$ with $\mathcal{D} \in \mathbf{ALG}$ be a lambda structure. Then the following statements are equivalent.*

(1) *$\mathcal{D} \models (\lambda x.M)N = M[x:=N]$, for all $M, N \in \Lambda$;*

(2) *$[\![\lambda x.M]\!]_\rho.d = [\![M]\!]_{\rho(x:=d)}$, for all $M \in \Lambda$ and $d \in \mathcal{D}$;*

(3) *$\mathcal{D}$ is a λ-model;*

(4) *$\mathcal{D} \models \{M = N \mid \lambda\boldsymbol{\beta} \vdash M = N\}$.*

(ii) *Let $\langle \mathtt{D}, F, G\rangle$ with $\mathcal{D} \in \mathbf{ALG}$ be a strict lambda structure. Then the following statements are equivalent.*

(1) *$\mathcal{D} \models (\lambda x.M)N = M[x:=N]$, for all $M, N \in \Lambda$ with $x \in \mathrm{FV}(M)$;*

(2) *$[\![\lambda x.M]\!]_\rho.d = [\![M]\!]_{\rho(x:=d)}$, for all $M \in \Lambda$, with with $x \in \mathrm{FV}(M)$, and $d \in \mathcal{D}$;*

(3) *$\mathcal{D}$ is a $\lambda\mathsf{I}$-model;*

(4) *$\mathcal{D} \models \{M = N \mid \lambda\boldsymbol{\beta}\mathsf{I} \vdash M = N\}$.*

PROOF. (i) (1)⇒(2). By (1) one has $[\![(\lambda x.M)N]\!]_\rho = [\![M[x:=N]]\!]_\rho$. Taking $N \equiv x$ and $\rho' = \rho(x:=d)$ one obtains

$$[\![(\lambda x.M)x]\!]_{\rho'} = [\![M]\!]_{\rho'},$$

hence

$$[\![\lambda x.M]\!]_\rho \cdot d = [\![M]\!]_{\rho'},$$

as $\rho \restriction \mathrm{FV}(\lambda x.M) = \rho' \restriction \mathrm{FV}(\lambda x.M)$.

(2)⇒(3). By (ii), Definition 18.1.4 and Proposition 18.1.5 all conditions to be a λ-model, see Definition 18.1.2, are fulfilled.

(3)⇒(4). By Theorem 5.3.4 in Barendregt [1984].

(4)⇒(1). Trivial.

(ii) Similarly. ∎

18.1.7. COROLLARY. *Let $\mathcal{D}_{F,G} = \langle \mathsf{D}, F, G\rangle$ be a (strict) lambda structure and a $\lambda(\mathsf{I})$-model. Then*

$$\mathcal{D} \text{ is a } \lambda(\mathsf{I})\boldsymbol{\eta}\text{-model} \Leftrightarrow \mathcal{D} \text{ is an extensional } \lambda(\mathsf{I})\text{-model.}$$

PROOF. (⇒) Suppose that for some ρ one has for all $d \in \mathcal{D}$

$$[\![Mx]\!]_{\rho[x:=d]} = [\![Nx]\!]_{\rho[x:=d]}.$$

Then by ($\boldsymbol{\eta}$) and Proposition 18.1.5(ii) one has

$$[\![M]\!]_\rho = [\![\lambda x.Mx]\!]_\rho = [\![\lambda x.Nx]\!]_\rho = [\![N]\!]_\rho.$$

(⇐) Note that by ($\boldsymbol{\beta}(\mathsf{I})$) one has $\mathcal{D} \models (\lambda x.Mx)y = My$, where x is fresh. Hence by extensionality one has $\mathcal{D} \models \lambda x.Mx = M$. ∎

Isomorphisms of λ-models

18.1.8. DEFINITION. Let $\mathcal{D}_{F,G}$ be a lambda structure.

(i) $\mathcal{D}_{F,G}$ is called *reflexive* if $F \circ G = \mathsf{Id}_{[\mathsf{D} \to \mathsf{D}]}$.

(ii) $\mathcal{D}_{F,G}$ is called *extensional* if $G \circ F = \mathsf{Id}_{\mathsf{D}}$.

18.1.9. PROPOSITION. *Let $\mathcal{D}_{F,G}$ be a lambda structure.*

(i) *If $\mathcal{D}_{F,G}$ is reflexive, then it is a λ-model.*

(ii) *If $\mathcal{D}_{F,G}$ is moreover extensional, then it is an extensional λ-model.*

PROOF. This is Theorem 5.4.4 of Barendregt [1984].

18.1.10. DEFINITION. (i) An isomorphism between two reflexive structures $\langle \mathcal{D}, F, G\rangle$ and $\langle \mathcal{D}', F', G'\rangle$ is a bijective mapping m such that

(1) $\mathsf{m}(G(f)) = G'(\mathsf{m} \circ f \circ \mathsf{m}^{-1})$

(2) $\mathsf{m}(F(d)(e)) = F'(\mathsf{m}(d))(\mathsf{m}(e))$

If we write $f^{\mathsf{m}} = \mathsf{m} \circ f \circ \mathsf{m}^{-1}$ then we can write these conditions as

$$\begin{array}{rcl} \mathsf{m}(G(f)) & = & G'(f^{\mathsf{m}}) \\ \mathsf{m}(d \cdot_F e) & = & \mathsf{m}(d) \cdot_{F'} \mathsf{m}(e). \end{array}$$

18.1.11. PROPOSITION. (i) *If $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic λ-models via* m *then for all λ-terms M and environments ρ:*

$$\mathsf{m}([\![M]\!]^{\mathcal{D}}_{\rho}) = [\![M]\!]^{\mathcal{D}'}_{\mathsf{m}\circ\rho}$$

(ii) *If two λ-models are isomorphic then they equal the same terms, i.e. $\mathcal{D} \models M = N$ iff $\mathcal{D}' \models M = N$.*

PROOF. (i) By induction on M.
(ii) By (i). ■

18.2. Filter models

Now we introduce the fundamental notion of filter structure, which will be used extensively in this Section. It is of paramount importance, and one can say that all the preceding sections in this Chapter are a build-up to it. Since the seminal paper Barendregt et al. [1983], this notion has played a major role in the study of the mathematical semantics of lambda calculus.

Remember Definition 15.4.2(ii) where for $\mathcal{T} \in \mathrm{TT}^{\top}$ and X a non-empty subset of $\mathcal{T}$ one defines the filter generated by X

$$\begin{array}{rcll} \uparrow X & = & \{x \in \mathcal{T} \mid \exists n \geq 1\, \exists x_1 \dots x_n \in X. x_1 \cap \dots \cap x_n \leq x\}, & \text{if } X \neq \emptyset; \\ \uparrow\emptyset & = & \{\top\}, & \text{else.} \end{array}$$

Now we extend this notion as follows.
18.2.1. DEFINITION. (i) Let $\mathcal{T} \in \mathrm{TT}$. Then we define $\uparrow^s X \in \mathcal{F}^{\mathcal{S}}_s$ by

$$\begin{array}{rcll} \uparrow^s X & = & \uparrow X, & \text{if } X \neq \emptyset; \\ \uparrow^s \emptyset & = & \emptyset. & \end{array}$$

18.2.2. DEFINITION. (i) Let $\mathcal{T} \in \mathrm{TT}^{\top}$. Define

$$\begin{array}{rcll} F^{\mathcal{T}} & : & [\mathcal{F}^{\mathcal{T}} \to [\mathcal{F}^{\mathcal{T}} \to \mathcal{F}^{\mathcal{T}}]], & \text{and} \\ G^{\mathcal{T}} & : & [[\mathcal{F}^{\mathcal{T}} \to \mathcal{F}^{\mathcal{T}}] \to \mathcal{F}^{\mathcal{T}}] & \end{array}$$

as follows.

$$\begin{array}{rcl} F^{\mathcal{T}}(X)(Y) & = & \uparrow\{B \in \mathbb{T}^{\mathcal{T}} \mid \exists A \in Y.(A \to B) \in X\}; \\ G^{\mathcal{T}}(f) & = & \uparrow\{A \to B \mid B \in f(\uparrow A)\}. \end{array}$$

(ii) Let $\mathcal{T} \in \mathrm{TT}$. Define

$$\begin{array}{lll} F_s^{\mathcal{T}} & : & [\mathcal{F}_s^{\mathcal{T}} \to_s [\mathcal{F}_s^{\mathcal{T}} \to_s \mathcal{F}_s^{\mathcal{T}}]], \qquad \text{and} \\ G_s^{\mathcal{T}} & : & [[\mathcal{F}_s^{\mathcal{T}} \to_s \mathcal{F}_s^{\mathcal{T}}] \to_s \mathcal{F}_s^{\mathcal{T}}] \end{array}$$

as follows.

$$\begin{array}{rcl} F_s^{\mathcal{T}}(X)(Y) & = & \uparrow^s \{B \in \mathbb{T}^{\mathcal{T}} \mid \exists A \in Y.(A \to B) \in X\}; \\ G_s^{\mathcal{T}}(f) & = & \uparrow^s \{A \to B \mid B \in f(\uparrow A)\}. \end{array}$$

18.2.3. Lemma. (i) *Let $TT \in \mathrm{TT}^{\top}$. Then $\langle \mathcal{F}^{\mathcal{T}}, F^{\mathcal{T}}, G^{\mathcal{T}} \rangle$ is a lambda structure.*
(ii) *Let $\mathcal{T} \in \mathrm{TT}$. Then $\mathcal{F}_s^{\mathcal{T}} = \langle \mathcal{F}_s^{\mathcal{T}}, F_s^{\mathcal{T}}, G_s^{\mathcal{T}} \rangle$ is a strict lambda structure.*

Proof. (i) It is easy to verify that $F^{\mathcal{T}}, G^{\mathcal{T}}$ are continuous.
(ii) Similarly. ■

18.2.4. Definition. (i) Let $\mathcal{T} \in \mathrm{TT}^{\top}$. Then $\mathcal{F}^{\mathcal{T}} = \langle \mathcal{F}^{\mathcal{T}}, F^{\mathcal{T}}, G^{\mathcal{T}} \rangle$ is called the *filter lambda structure* over $\mathcal{T}$.
(ii) Let $\mathcal{T} \in \mathrm{TT}$. Then $\mathcal{F}_s^{\mathcal{T}} = \langle \mathcal{F}_s^{\mathcal{T}}, F_s^{\mathcal{T}}, G_s^{\mathcal{T}} \rangle$ is called the *strict filter lambda structure* over $\mathcal{T}$.

Recall that by Proposition 15.3.3 a compatible element of $\mathrm{TT}^{\top}$ induces a type structure in $\mathrm{TS}^{\top}$. We can take advantage in this case of the equivalencies between type and zip structures (Theorems 17.3.17 and 17.3.37).

18.2.5. Lemma. (i) *If $\mathcal{S} \in \mathbf{TS}^{\top}$, then $F^{\mathcal{S}} = F_{Z^{\mathcal{S}}}$ and $G^{\mathcal{S}} = G_{Z^{\mathcal{S}}}$, where $F_{Z^{\mathcal{S}}}$ and $G_{Z^{\mathcal{S}}}$ are defined in Definitions 17.3.10 and 17.4.12.*
(ii) *If $\mathcal{S} \in \mathbf{TS}$, then $F_s^{\mathcal{S}} = F_{Z_s^{\mathcal{S}}}$ and $G_s^{\mathcal{S}} = G_{Z_s^{\mathcal{S}}}$, where $F_{Z_s^{\mathcal{S}}}$ and $G_{Z_s^{\mathcal{S}}}$ are defined in Definitions 17.3.26 and 17.4.32*

Proof. (i) Taking the suprema in $\mathcal{F}^{\mathcal{S}}$ one has

$$\begin{array}{rcl} F^{\mathcal{S}}(X)(Y) & = & \uparrow\{\uparrow A \mid \exists B \in Y.(B \to A) \in X\} \\ & = & \bigsqcup\{\uparrow A \mid \exists B \in Y.\uparrow(B \to A) \subseteq X\} \\ & = & \bigsqcup\{\uparrow A \mid \exists \uparrow B \subseteq Y.Z^{\mathcal{S}}(\uparrow B, \uparrow A) \subseteq X\} \\ & = & X \cdot_{Z^{\mathcal{S}}} Y. \end{array}$$

Moreover,

$$\begin{array}{rcl} G^{\mathcal{S}}(f) & = & \uparrow\{B \to A \mid A \in f(\uparrow B)\} \\ & = & \bigsqcup\{\uparrow(B \to A) \mid A \in f(\uparrow B)\} \\ & = & \bigsqcup\{Z(\uparrow B, \uparrow A) \mid \uparrow A \subseteq f(\uparrow B)\}. \end{array}$$

(ii) Now the suprema are taken in $\mathcal{F}_s^{\mathcal{S}}$ and $\bigsqcup \emptyset = \emptyset$, the bottom of $\mathcal{F}_s^{\mathcal{S}}$. ■

Now we work towards the characterization of those type theories $\mathcal{T}$ such that $\mathcal{F}^{\mathcal{T}}$ is a $\lambda(\mathsf{I})$-model, a so-called filter λ-model. This happens in 18.2.6-18.2.14.

The following *type-semantics theorem* is important. It has as consequence that for a closed untyped lambda term M and a $\mathcal{T} \in \mathrm{TT}^{\top}$ one has

$$[\![M]\!]^{\mathcal{F}^{\mathcal{T}}} = \{A \mid \vdash^{\mathcal{T}}_{\cap^{\top}} M : A\},$$

i.e. the semantical meaning of M in the filter λ-model corresponding to a $\mathcal{T} \in \mathrm{TT}^{\top}$ is the collection of its types. For a $\mathcal{T} \in \mathrm{TT}$ one has

$$[\![M]\!]^{\mathcal{F}^{\mathcal{T}}_{s}} = \{A \mid \vdash^{\mathcal{T}}_{\cap} M : A\}.$$

18.2.6. Definition. A context Γ *agrees with an environment* $\rho \in \mathsf{Env}^{(s)}_{\mathcal{F}^{\mathcal{T}}_{(s)}}$, notation $\Gamma \models \rho$, if

$$(x : A) \in \Gamma \;\Rightarrow\; A \in \rho(x).$$

18.2.7. Proposition. (i) $\Gamma \models \rho \;\&\; \Gamma' \models \rho \;\Rightarrow\; \Gamma \uplus \Gamma' \models \rho$.
(ii) $\Gamma \models \rho[x := \uparrow^{(s)} A] \;\Rightarrow\; \Gamma \backslash x \models \rho$.

Proof. Immediate. ■

18.2.8. Theorem (Type-semantics Theorem). (i) *Let $\mathcal{T} \in \mathrm{TT}^{\top}$ and $\mathcal{F}^{\mathcal{T}}$ its corresponding filter structure. Then, for any lambda-term M and $\rho \in \mathsf{Env}_{\mathcal{F}^{\mathcal{T}}}$,*

$$[\![M]\!]^{\mathcal{F}^{\mathcal{T}}}_{\rho} = \{A \mid \Gamma \vdash^{\mathcal{T}}_{\cap^{\top}} M : A \text{ for some } \Gamma \models \rho\}.$$

(ii) *Let $\mathcal{T} \in \mathrm{TT}$ and $\mathcal{F}^{\mathcal{T}}_{s}$ its corresponding strict filter structure. Then, for any lambda-term M and $\rho \in \mathsf{Env}^{s}_{\mathcal{F}^{\mathcal{T}}_{s}}$,*

$$[\![M]\!]^{\mathcal{F}^{\mathcal{T}}_{s}}_{\rho} = \{A \mid \Gamma \vdash^{\mathcal{T}}_{\cap} M : A \text{ for some } \Gamma \models \rho\}.$$

Proof. (i) By induction on the structure of M.
If $M \equiv x$, then

$$\begin{aligned}
[\![x]\!]^{\mathcal{F}^{\mathcal{T}}}_{\rho} &= \rho(x) \\
&= \{A \mid A \in \rho(x)\} \\
&= \{A \mid A \in \rho(x) \;\&\; x : A \vdash^{\mathcal{T}}_{\cap^{\top}} x : A\} \\
&= \{A \mid \Gamma \vdash^{\mathcal{T}}_{\cap^{\top}} x : A \text{ for some } \Gamma \models \rho\}, && \text{by Definition 18.2.6 and the Inversion Theorem 16.1.1(i).}
\end{aligned}$$

If $M \equiv NL$, then

$$
\begin{aligned}
[\![NL]\!]^{\mathcal{T}}_{\rho} &= [\![N]\!]^{\mathcal{T}}_{\rho} \cdot [\![L]\!]^{\mathcal{T}}_{\rho} \\
&= \uparrow\{A \mid \exists B \in [\![L]\!]^{\mathcal{T}}_{\rho}.(B \to A) \in [\![N]\!]^{\mathcal{T}}_{\rho}\} \\
&= \{A \mid \exists k{>}0 \exists B_1, \ldots, B_k, C_1, \ldots, C_k. \\
&\quad [\,(B_i{\to}C_i) \in [\![N]\!]^{\mathcal{T}}_{\rho} \ \&\ B_i \in [\![L]\!]^{\mathcal{T}}_{\rho} \ \&\ (\textstyle\bigcap_{1 \le i \le k} C_i) \le A]\} \cup \uparrow\{\top\}, \\
&\quad \text{by definition of } \uparrow, \\
&= \{A \mid \exists k{>}0 \exists B_1, \ldots, B_k, C_1, \ldots, C_k, \exists \Gamma_{1i}, \Gamma_{2i} \\
&\quad [\Gamma_{1i}, \Gamma_{2i} \models \rho \ \&\ \Gamma_{1i} \vdash^{\mathcal{T}}_{\cap\top} N : (B_i{\to}C_i) \\
&\quad \ \&\ \Gamma_{2i} \vdash^{\mathcal{T}}_{\cap\top} L : B_i \ \&\ C_1 \cap \ldots \cap C_k \le A]\} \cup \uparrow\{\top\}, \\
&\quad \text{by the induction hypothesis,} \\
&= \{A \mid \Gamma \vdash^{\mathcal{T}}_{\cap\top} NL : A \text{ for some } \Gamma \models \rho\}, \\
&\quad \text{taking } \Gamma = \Gamma_{11} \uplus \ldots \uplus \Gamma 1k \uplus \ldots \uplus \Gamma_{21} \uplus \ldots \uplus \Gamma_{2k}, \\
&\quad \text{by Theorem 16.1.1(ii) and Proposition 18.2.7(i).}
\end{aligned}
$$

If $M \equiv \lambda x.N$, then

$$
\begin{aligned}
[\![\lambda x.N]\!]^{\mathcal{T}}_{\rho} &= G^{\mathcal{T}}(\boldsymbol{\lambda} X \in \mathcal{F}^{\mathcal{T}}.[\![N]\!]^{\mathcal{T}}_{\rho[x:=X]}) \\
&= \uparrow\{(B{\to}C) \mid C \in [\![N]\!]^{\mathcal{T}}_{\rho[x:=\uparrow B]}\} \\
&= \{A \mid \exists k{>}0 \exists B_1, \ldots, B_k, C_1, \ldots, C_k. \exists \Gamma_i [\Gamma_i \models \rho[x:=\uparrow B_i] \ \& \\
&\quad \Gamma_i, x{:}B_i \vdash^{\mathcal{T}}_{\cap\top} N : C_i \ \&\ (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \le A], \\
&\quad \text{by the induction hypothesis,} \\
&= \{A \mid \Gamma \vdash^{\mathcal{T}}_{\cap\top} \lambda x.N : A \text{ for some } \Gamma \models \rho\}, \\
&\quad \text{taking } \Gamma = (\Gamma_1 \uplus \ldots \uplus \Gamma_k)\backslash x, \text{ by Theorem 16.1.1(iii), rule } (\le) \\
&\quad \text{and Proposition 18.2.7(ii).}
\end{aligned}
$$

(ii) Similarly, with $\uparrow$ replaced by $\uparrow^s$. Note that in the case $M = NL$ we drop '$\cup\uparrow\{\top\}$' both times. In case $M = \lambda x.N$, using Definition 18.1.4, it follows that $[\![\lambda x.N]\!]^{\mathcal{T}}_{\rho} = \uparrow^s \{(B{\to}C) \mid C \in [\![N]\!]^{\mathcal{T}}\}_{\rho[x:=\uparrow B]}$ holds, because $\uparrow B \neq \emptyset$. ∎

18.2.9. Corollary. (i) *Let* $\mathcal{T} \in \mathrm{TT}^{\top}$. *Then*

$$\mathcal{F}^{\mathcal{T}} \textit{ is a } \lambda\textit{-model} \Leftrightarrow [\Gamma \vdash^{\mathcal{T}}_{\cap\top} (\lambda x.M) : (B{\to}A) \ \Rightarrow\ \Gamma, x{:}B \vdash^{\mathcal{T}}_{\cap\top} M : A].$$

(ii) *Let* $\mathcal{T} \in \mathrm{TT}$. *Then*

$$\mathcal{F}^{\mathcal{T}}_s \textit{ is a } \lambda\mathsf{I}\textit{-model} \Leftrightarrow$$
$$[\Gamma \vdash^{\mathcal{T}}_{\cap} (\lambda x.M) : (B{\to}A) \ \&\ x \in \mathrm{FV}(M) \ \Rightarrow\ \Gamma, x{:}B \vdash^{\mathcal{T}}_{\cap} M : A].$$

PROOF. (i) By Propositions 18.1.6(i), 16.2.1(ii) and Corollary 16.2.5(i).
(ii) By Propositions 18.1.6(ii), 16.2.1(i) and Corollary 16.2.5(ii). ■

18.2.10. COROLLARY. (i) *Let* $\mathcal{T} \in \mathrm{TT}^{\top}$. *Then*

$$\mathcal{T} \text{ is } \boldsymbol{\beta}\text{-sound} \Rightarrow \mathcal{F}^{\mathcal{T}} \text{ is a } \lambda\text{-model.}$$

(ii) *Let* $\mathcal{T} \in \mathrm{TT}$. *Then*

$$\mathcal{T} \text{ is } \boldsymbol{\beta}\text{-sound} \Rightarrow \mathcal{F}_s^{\mathcal{T}} \text{ is a } \lambda\mathsf{I}\text{-model.}$$

PROOF. By the Corollary above and Theorem 16.1.10(iii). ■

18.2.11. COROLLARY. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{BCD}, \mathrm{AO}, \mathrm{Plotkin}, \mathrm{Engeler}, \mathrm{CDS}\}$. *Then*

$$\mathcal{F}^{\mathcal{T}} \text{ is a } \lambda\text{-model.}$$

(ii) *Let* $\mathcal{T} \in \{\mathrm{HL}, \mathrm{CDV}, \mathrm{CD}\}$. *Then*

$$\mathcal{F}_s^{\mathcal{T}} \text{ is a } \lambda\mathsf{I}\text{-model.}$$

PROOF. (i) By (i) of the previous Corollary and Theorem 16.1.8.
(ii) By (ii) of the Corollary, using Theorem 16.1.8. ■

18.2.12. PROPOSITION. (i) *Let* $\mathcal{T} \in \mathrm{TT}^{\top}$. *Then*

$$\mathcal{T} \text{ is natural and } \boldsymbol{\beta}\text{- and } \boldsymbol{\eta}^{\top}\text{-sound} \Rightarrow \mathcal{F}^{\mathcal{T}} \text{ is an extensional } \lambda\text{-model.}$$

(ii) *Let* $\mathcal{T} \in \mathrm{TT}$. *Then*

$$\mathcal{T} \text{ is proper and } \boldsymbol{\beta}\text{- and } \boldsymbol{\eta}\text{-sound} \Rightarrow \mathcal{F}_s^{\mathcal{T}} \text{ is an extensional } \lambda\mathsf{I}\text{-model.}$$

PROOF. (i) and (ii). $\mathcal{F}^{\mathcal{T}}$ ($\mathcal{F}_s^{\mathcal{T}}$) is a $\lambda(\mathsf{I})$-model by Corollary 18.2.10(i)((ii)). For extensionality by Corollary 18.1.9 one needs to verify for $x \notin \mathrm{FV}(M)$

$$[\![\lambda x.Mx]\!]_\rho = [\![M]\!]_\rho. \qquad (\boldsymbol{\eta})$$

This follows from Theorems 18.2.8(i), and 16.2.15(i). ■

18.2.13. COROLLARY. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}\}$. *Then*

$$\mathcal{F}^{\mathcal{T}} \text{ is an extensional } \lambda\text{-model.}$$

(ii) *Let* $\mathcal{T} = \mathrm{HL}$. *Then*

$$\mathcal{F}_s^{\mathcal{T}} \text{ is an extensional } \lambda\mathsf{I}\text{-model.}$$

PROOF. (i) and (ii) By Corollary 16.2.13. ■