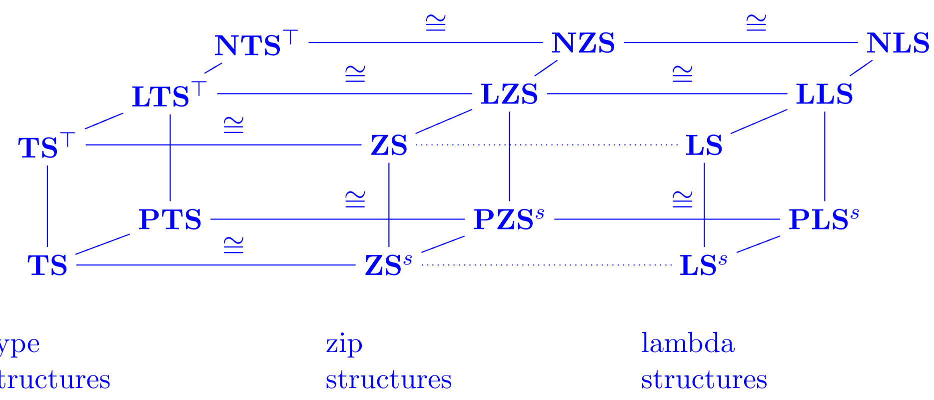

Figure 17.3: Equivalences proved in Sections 17.3 and 17.4

17.1. Meet semi-lattices and algebraic lattices

Categories of meet semi-lattices

Remember the following notions, see Definitions 15.3.8-15.3.10. The category **MSL** has as objects at most countable meet semi-lattices and as morphisms maps preserving $\leq$ and $\cap$.

The category $\mathbf{MSL}^\top$ is as **MSL**, but based on top meet semi-lattices. So now morphisms also should preserve the top.

The category **TS** has as objects the at most countable type structures and as morphisms maps $f : \mathcal{S} \to \mathcal{S}'$, preserving $\leq, \cap, \to$

The category $\mathbf{TS}^\top$ is as **TS**, but based on top type structures. Now also morphisms should preserve the top.

In Definition 15.3.10 we defined four full subcategories of **TS** by specifying in each case the objects: $\mathbf{GTS}^\top$ with as objects the graph top type structures; $\mathbf{LTS}^\top$ with as objects the lazy top type structures; $\mathbf{NTS}^\top$ with as objects the natural top type structures; **PTS** with as objects the proper type structures.

Categories of algebraic lattices

Comment:

- indexes are denoted either by I or by $\mathcal{I}$, you must choose to give some meaning to this notational difference (I finite and $\mathcal{I}$ possibly infinite?), state and respect it, or use the same notation.

The following has already been given in Definition 14.2.1, but now we treat in in greater detail.

17.1.1. Definition. (i) A *complete lattice* is a poset $\mathcal{D} = (\mathtt{D}, \sqsubseteq)$ such that for arbitrary $X \subseteq \mathtt{D}$ the supremum $\bigsqcup X \in \mathtt{D}$ exists. Then one has also a *top* element $\top_{\mathcal{D}} = \bigsqcup \mathtt{D}$, a *bottom* element $\bot_{\mathcal{D}} = \bigsqcup \emptyset$, arbitrary *infima*

$$\bigsqcap X = \bigsqcup \{z \mid \forall x \in X. z \sqsubseteq x\}$$

and the *sup* and *inf* of two elements

$$x \sqcup y = \bigsqcup\{x, y\}, \quad x \sqcap y = \bigsqcap\{x, y\}.$$

(ii) A subset $Z \subseteq \mathtt{D}$ is called *directed* if Z is non-empty and

$$\forall x, y \in Z\ \exists z \in Z. x \sqsubseteq z\ \&\ y \sqsubseteq z.$$

(iii) An element $d \in \mathtt{D}$ is called *compact* (also sometimes called *finite* in the literature) if for every directed $Z \subseteq \mathtt{D}$ one has

$$d \sqsubseteq \bigsqcup Z \ \Rightarrow\ \exists z \in Z. d \sqsubseteq z.$$

Note that if d, e are compact, then so is $d \sqcup e$[3].
(iv) $\mathcal{K}(\mathcal{D}) = \{d \in \mathtt{D} \mid d \text{ is compact}\}$.
(v) $\mathcal{K}^s(\mathcal{D}) = \mathcal{K}(\mathcal{D}) - \{\bot_{\mathcal{D}}\}$.
(vi) $\mathcal{D}$ is called an *algebraic lattice* if

$$\forall x \in \mathtt{D}. x = \bigsqcup\{e \in \mathcal{K}(\mathcal{D}) \mid e \sqsubseteq x\}.$$

$\mathcal{D}$ is called an ω-*algebraic lattice* if moreover $\mathcal{K}(\mathcal{D})$ is countable (finite or countably infinite).

Instead of $d \in \mathtt{D}$ or $X \subseteq \mathtt{D}$ we often write $d \in \mathcal{D}$ or $X \subseteq \mathcal{D}$, respectively. When useful we will decorate $\sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top, \sqcup$ and $\sqcap$ with $\mathcal{D}$, e.g. $\sqsubseteq_{\mathcal{D}}$ etcetera.

The following connects the notion of a compact element to the notion of compact subset of a topological space.

17.1.2. Lemma. *Let $\mathcal{D}$ be a complete lattice. Then $d \in \mathcal{D}$ is compact iff*

$$\forall Z \subseteq \mathcal{D}. [d \sqsubseteq Z \ \Rightarrow\ \exists Z_0 \subseteq Z. [Z_0 \textit{ is finite } \& \ d \sqsubseteq \bigsqcup Z_0]].$$

Proof. ($\Rightarrow$) Suppose $d \in \mathcal{D}$ is compact. Given $Z \subseteq \mathcal{D}$, let

$$Z^+ = \{\bigsqcup Z_0 \mid Z_0 \subseteq Z\ \&\ Z_0 \text{ finite}\}.$$

Then $Z \subseteq Z^+, \bigsqcup Z_0 = \bigsqcup Z$ and Z^+ is directed. Hence

$$\begin{array}{rcl} d \sqsubseteq \bigsqcup Z & \Rightarrow & d \sqsubseteq \bigsqcup Z^+ \\ & \Rightarrow & \exists z^+ \in Z^+. d \sqsubseteq z^+ \\ & \Rightarrow & \exists Z_0 \subseteq Z. d \sqsubseteq \bigsqcup Z_0\ \&\ Z_0 \text{ is finite.} \end{array}$$

($\Leftarrow$) Suppose $d \sqsubseteq \bigsqcup Z$ with $Z \subseteq \mathcal{D}$ directed. By the condition $d \sqsubseteq \bigsqcup Z_0$ for some finite $Z_0 \subseteq Z$. If Z_0 is non-empty, then by the directedness of Z there exists a $z \in Z$ such that $z \sqsupseteq \bigsqcup Z_0 \sqsupseteq d$. If Z_0 is empty, then $d = \bot$ and we can take an arbitrary element z in the non-empty Z satisfying $d \sqsubseteq z$. ∎

[3] In general it is not true that if $d \sqsubseteq e \in \mathcal{K}(\mathcal{D})$, then $d \in \mathcal{K}(\mathcal{D})$; take for example $\omega + 1$ in the ordinal $\omega + \omega = \{0, 1, 2, \ldots\ \omega, \omega + 1, \omega + 2, \ldots\}$. It is compact, but $\omega\ \ (\sqsubseteq \omega + 1)$ is not.

17.1.3. Notation. Let $\mathcal{D}$ be an ω-algebraic lattice. For $x\in\mathcal{D}$, write

$$K(x) = \{d\in\mathcal{K}(\mathcal{D}) \mid d \sqsubseteq x\}.$$

In this Chapter $a, b, c, d\ldots$ always denote compact elements in lattices. Generic elements are denoted by $x, y, z\ldots$. **Comment:** I do not agree to cancel this, it helps the reader! Henk: But we are not consistent: in 17.1.9 a' is not compact. Also not in 17.1.7(ii).

17.1.4. Definition. Let $\mathcal{D}, \mathcal{E}$ be complete lattices and $f : \mathcal{D}\to\mathcal{E}$.
(i) f is called *(Scott) continuous* iff for all directed $X \subseteq \mathcal{D}$ one has

$$f(\bigsqcup X) = \bigsqcup f(X)\ (= \bigsqcup\{f(x) \mid x\in X\}).$$

(ii) $[\mathcal{D}\to\mathcal{E}] = \{f : \mathcal{D}\to\mathcal{E} \mid f$ is Scott continuous functions$\}$.
(iii) f is called *strict* iff $f(\bot) = \bot$.
(iv) Write $[\mathcal{D}\to_{\rm s}\mathcal{E}]$ for the collection of continuous strict maps.

17.1.5. Proposition. *Let $\mathcal{D}, \mathcal{E}$ be algebraic lattices.*
(i) *Let $f\in[\mathcal{D}\to\mathcal{E}]$. Then for $x\in\mathcal{D}$*

$$f(x) = \bigsqcup\{f(e) \mid e \sqsubseteq x \ \&\ e\in\mathcal{K}(\mathcal{D})\}.$$

(ii) *Let $f, g\in[\mathcal{D}\to\mathcal{E}]$. Suppose $f\restriction\mathcal{K}(\mathcal{D}) = g\restriction\mathcal{K}(\mathcal{D})$. Then $f = g$.*

Proof. (i) Use that $x = \bigsqcup\{e \mid e \sqsubseteq x\}$ is a directed sup and that f is continuous.
(ii) By (i). ■

17.1.6. Definition. The category **ALG** has as objects the ω-algebraic complete lattices and as morphisms the continuous maps.

17.1.7. Definition. (i) $[\mathcal{D}\to\mathcal{D}']$ is partially ordered pointwise as follows.

$$f \sqsubseteq g \ \Leftrightarrow\ \forall x\in\mathcal{D}.f(x) \sqsubseteq g(x).$$

(ii) If $a\in\mathcal{D}$, $a'\in\mathcal{D}'$, then $a{\mapsto}a'$ is the *step function* defined by

$$\begin{array}{rcll}(a{\mapsto}a')(d) &=& a', & \text{if } a \sqsubseteq d;\\ &=& \bot_{\mathcal{D}'}, & \text{else.}\end{array}$$

17.1.8. Lemma. *$[\mathcal{D}\to\mathcal{D}']$ is a complete lattice with*

$$(\bigsqcup_{f\in X} f)(d) = \bigsqcup_{f\in X} f(d).$$

17.1.9. Lemma. *For $a, b\in\mathcal{D}$, $a', b'\in\mathcal{D}'$ and $f\in[\mathcal{D}\to\mathcal{D}']$ one has*
(i) *a compact $\Rightarrow$ $a{\mapsto}a'$ is continuous.*
(ii) *$a{\mapsto}a'$ is continuous and $a' \neq \bot$ $\Rightarrow$ a is compact.*
(iii) *a' compact $\Leftrightarrow$ $a{\mapsto}a'$ compact.*
(iv) *$a' \sqsubseteq f(a)$ $\Leftrightarrow$ $(a{\mapsto}a') \sqsubseteq f$.*

(v) $b \sqsubseteq a \;\&\; a' \sqsubseteq b' \;\Rightarrow\; (a{\mapsto}a') \sqsubseteq (b{\mapsto}b')$.
(vi) $(a{\mapsto}a') \sqcup (b{\mapsto}b') \sqsubseteq (a \sqcap b){\mapsto}(a' \sqcup b')$.

Proof. Easy. ■

17.1.10. Lemma. *For all* $b, a_1, \ldots, a_n \in \mathcal{D}$, $b', a'_1, \ldots, a'_n \in \mathcal{D}'$, *and* $f \in [\mathcal{D}{\to}\mathcal{D}']$

$$(b{\mapsto}b') \sqsubseteq (a_1{\mapsto}a'_1) \sqcup \ldots \sqcup (a_n{\mapsto}a'_n) \;\Leftrightarrow$$
$$\Leftrightarrow\; \exists I{\subseteq}\{1, \ldots, n\}\, [\sqcup_{i \in I} a_i \sqsubseteq b \;\&\; b' \sqsubseteq \sqcup_{i \in I} a'_i].$$

Clearly in $(\Rightarrow)$ *we have* $I \neq \emptyset$ *if* $d \neq \bot_{\mathcal{D}}$.

Proof. Easy. ■

17.1.11. Proposition. *Let* $\mathcal{D}, \mathcal{D}' \in \mathbf{ALG}$.
(i) *For* $f \in [\mathcal{D}{\to}\mathcal{D}']$ *one has* $f = \bigsqcup\{a \Rightarrow a' \mid a' \sqsubseteq f(a),\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')\}$.
(ii) *Let* $\mathcal{D} \in \mathbf{ALG}$ *and let* $f : [\mathcal{D}{\to}\mathcal{D}']$ *be compact. Then*

$$f = (a_1{\mapsto}a'_1) \sqcup \ldots \sqcup (a_n{\mapsto}a'_n),$$

for some $a_1, \ldots, a_n \in \mathcal{K}(\mathcal{D}), a'_1, \ldots, a'_n \in \mathcal{K}(\mathcal{D}')$.
(iii) $[\mathcal{D}{\to}\mathcal{D}'] \in \mathbf{ALG}$.

Proof. (i) It suffices to show that RHS and LHS are equal when applied to an arbitrary element $d \in \mathcal{D}$.

$$\begin{array}{rcl}
f(d) & = & f(\bigsqcup\{a \mid a \sqsubseteq d \;\&\; a \in \mathcal{K}(\mathcal{D})\}) \\
& = & \bigsqcup\{f(a) \mid a \sqsubseteq d \;\&\; a \in \mathcal{K}(\mathcal{D})\} \\
& = & \bigsqcup\{a' \mid a' \sqsubseteq f(a) \;\&\; a \sqsubseteq d \;\&\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')\} \\
& = & \bigsqcup\{(a{\mapsto}a')(d) \mid a' \sqsubseteq f(a) \;\&\; a \sqsubseteq d \;\&\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')\} \\
& = & \bigsqcup\{(a{\mapsto}a')(d) \mid a' \sqsubseteq f(a) \;\&\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')\} \\
& = & (\bigsqcup\{(a{\mapsto}a') \mid a' \sqsubseteq f(a) \;\&\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')(d)\}.
\end{array}$$

(ii) For f compact one has $f = \bigsqcup\{a{\mapsto}a' \mid a' \sqsubseteq f(a) \;\&\; a \in \mathcal{K}(\mathcal{D}), a' \in \mathcal{K}(\mathcal{D}')\}$, by (i). Hence by Lemma 17.1.2 for some $a_1, \ldots, a_n \in \mathcal{K}(\mathcal{D}), a'_1, \ldots, a'_n \in \mathcal{K}(\mathcal{D}')$

$$f = (a_1{\mapsto}a'_1) \sqcup \ldots \sqcup (a_n{\mapsto}a'_n). \tag{17.1}$$

(iii) It remains to show that there are only countably many compact elements in $[\mathcal{D}{\to}\mathcal{D}]$. Since $\mathcal{K}(\mathcal{D})$ is countable, there are only countably many expressions in the RHS of (17.1). (The cardinality is $\leq \Sigma_n n.\aleph_0^2 = \aleph_0$.) Therefore there are countable many compact $f \in [\mathcal{D}{\to}\mathcal{D}]$. (There may be more expressions on the RHS for one f, but this results in less compact elements.) ■

17.1.12. Definition. (i) The category $\mathbf{ALG}_a$ has the same objects as $\mathbf{ALG}$ and as morphisms $\mathbf{ALG}_a(\mathcal{D}, \mathcal{D}')$ maps $f : \mathcal{D}{\to}\mathcal{D}'$ that satisfy the properties 'compactness preserving' and 'additive':

(cmp-pres)	$\forall a \in \mathcal{K}(\mathcal{D}).f(a) \in \mathcal{K}(\mathcal{D}')$;
(add)	$\forall X \subseteq \mathcal{D}.f(\bigsqcup X) = \bigsqcup f(X)$.