Chapter 15

The Systems $\lambda_\cap^\mathcal{T}$ and $\lambda_{\cap^\top}^\mathcal{T}$ 31.10.2006:581

Intersection types are syntactic objects forming a free algebra $\mathbb{T}$, which is generated from a set of atoms $\mathbb{A}$, using the operators $\to$ and $\cap$. Postulating axioms and rules an *intersection type theory* results, which characterizes a preorder $\leq_\mathcal{T}$ on $\mathbb{T}$ with $\cap$ as set intersection, giving for two elements a greatest lower bound (glb). The class of these theories is abbreviated[1] as TT.

Taking into account the intuitive meaning of $\to$ as function space constructor one usually requires that the resulting equivalence relation $=_\mathcal{T}$ is a congruence. Then we speak of a *compatible* type theory, having a corresponding *type structure*

$$\langle \mathcal{S}, \leq, \cap, \to \rangle = \langle \mathbb{T}/{=_\mathcal{T}}, \leq, \cap, \to \rangle.$$

The collection of type structures is denoted by TS. Each type structure can be seen as coming from a compatible type theory and compatible type theories and type structures are basically the same. In the present Part III of this book both these syntactic and semantic aspects will be exploited.

$\mathrm{TT}^\top$ is a subset of TT, the set of *top type theories*, where the set of atoms $\mathbb{A}$ has a top element $\top$. Similarly a top intersection type structure $\mathrm{TS}^\top$ is of the form $\langle \mathcal{S}, \leq, \cap, \to, \top \rangle$.

The various type theories (and type structures) are introduced together in order to give reasonably uniform proofs of their properties as well of those of the corresponding type assignment systems and filter models.

Given a (top) type theory $\mathcal{T}$, one can define a corresponding type assignment system. These type assignment systems will be studied extensively in later chapters. We also introduce so-called *filters*, sets of types closed under intersection $\cap$ and preorder $\leq$. These play an important role in Chapter 17 to establish equivalences of categories and in Chapter 18 to build λ-models.

In Section 15.1 we define the notion of type theory and introduce 13 specific examples, including basic lemmas for these. In Section 15.2 the type assignment systems are defined. In Section 15.3 we discuss intersection type structures and introduce specific categories of lattices and type structures to accommodate these. Finally in Section 15.4 the filters are defined.

[1]Since all type theories in Part III of this book are using the intersection operator, we keep this implicit and often simply speak about *(top) type theories*, leaving 'intersection' implicit.

15.1. Type theories

As in Chapter 14 we will use as syntactic types $\mathbb{T} = \mathbb{T}_\cap^{\mathbb{A}}$ defined by

$$\mathbb{T} = \mathbb{A} \mid \mathbb{T}{\to}\mathbb{T} \mid \mathbb{T} \cap \mathbb{T}$$

as abstract syntax. This time we will use various sets of atoms $\mathbb{A}$. The letters $\alpha, \beta, \gamma, \ldots$ range over arbitrary atoms. If we need special atoms for a special purpose, like for example $\top, \omega, \varphi$, then we can identify them with some of the ψ_i, i.e. $\top = \psi_0,\ \omega = \psi_1,\ \varphi = \psi_2$.

15.1.1. Definition. (i) An *intersection type theory over a set of type atoms* $\mathbb{A}$ is a set of judgements $\mathcal{T}$ of the form $A \leq B$ (to be read: A is a subtype of B), with $A, B \in \mathbb{T}_\cap^{\mathbb{A}}$, satisfying the following axioms and rules.

(refl)	$A \leq A$
(incl$_L$)	$A \cap B \leq A$
(incl$_R$)	$A \cap B \leq B$
(glb)	$\dfrac{C \leq A \quad C \leq B}{C \leq A \cap B}$
(trans)	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$

This means that e.g. $(A \leq A) \in \mathcal{T}$ and $(A \leq B), (B \leq C) \in \mathcal{T} \ \Rightarrow\ (A \leq C) \in \mathcal{T}$, for all A, B, C.

(ii) A *top intersection type theory* is an intersection type theory with an element $\top \in \mathbb{T}$ for which one can derive

$(\top)$	$A \leq \top$

(iii) The notion '(top) intersection type theory' will be abbreviated as '(top) type theory', as the 'intersection' part is default.

(iv) TT stands for the set of type theories and $\mathrm{TT}^\top$ for that of top type theories.

(v) If $\mathcal{T} \in \mathrm{TT}^{(\top)}$ over $\mathbb{A}$, then we also write $\mathbb{T}^{\mathcal{T}}$ for $\mathbb{T}_\cap^{\mathbb{A}}$.

In this and the next section $\mathcal{T}$ ranges over elements of $\mathrm{TT}^{(\top)}$. Most of them have some extra axioms or rules, the above set being the minimum requirement. For example the theory BCD over $\mathbb{A} = \mathbb{A}_\infty^\top$, defined in Chapter 14 is a $\mathrm{TT}^\top$ and has the extra axioms $(\top{\to})$ and $({\to}\cap)$ and rule $({\to})$.

15.1.2. Notation. Let $\mathcal{T} \in \mathrm{TT}$. We write the following.

(i) $A \leq_{\mathcal{T}} B$ or $\vdash_{\mathcal{T}} A \leq B$ for $(A \leq B) \in \mathcal{T}$.

(ii) $A =_{\mathcal{T}} B$ for $A \leq_{\mathcal{T}} B \leq_{\mathcal{T}} A$.

(iii) $A <_{\mathcal{T}} B$ for $A \leq B \ \&\ A \neq_{\mathcal{T}} B$.

(iv) If there is little danger of confusion and $\mathcal{T}$ is clear from the context, then we will write $\leq, =, <$ for respectively $\leq_{\mathcal{T}}, =_{\mathcal{T}}, <_{\mathcal{T}}$.

(v) We write $A \equiv B$ for syntactic identity. E.g. $A \cap B \equiv A \cap B$, but $A \cap B \not\equiv B \cap A$.

15.1.3. Lemma. *For any $\mathcal{T}$ one has $A \cap B =_{\mathcal{T}} B \cap A$.*

Proof. By (incl_L), (incl_R) and (glb). ■

15.1.4. Definition. $\mathcal{T}$ is called *compatible* iff the following rule holds.

$$(\rightarrow^{=}) \quad \frac{A = A' \quad B = B'}{(A \rightarrow B) = (A' \rightarrow B')}$$

This means $A =_{\mathcal{T}} A'$ & $B =_{\mathcal{T}} B' \Rightarrow (A \rightarrow B) =_{\mathcal{T}} (A' \rightarrow B')$. One way to insure this is to adopt $(\rightarrow^{=})$ as rule determining $\mathcal{T}$.

15.1.5. Remarks. (i) Let $\mathcal{T}$ be compatible. Then by Lemma 15.1.3 one has

$$(A \cap B) \rightarrow C = (B \cap A) \rightarrow C.$$

(ii) The rule (glb) implies that the following rule is admissible.

$$(\text{mon}) \quad \frac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$$

A $\mathcal{T} \in \text{TT}$ can be seen as a structure with a pre-order

$$\mathcal{T} = \langle \mathbb{T}, \leq, \cap, \rightarrow \rangle.$$

This means that $\leq$ is reflexive and transitive, but not necessarily anti-symmetric

$$A \leq_{\mathcal{T}} B \;\&\; B \leq_{\mathcal{T}} A \not\Rightarrow A =_{\mathcal{T}} B.$$

If $\mathcal{T}$ is compatible one can go over to equivalence classes and obtain a type structure

$$\mathcal{T}/{=_{\mathcal{T}}} = \langle \mathbb{T}/{=_{\mathcal{T}}}, \leq, \cap, \rightarrow \rangle.$$

If moreover $\mathcal{T} \in \text{TT}^{\top}$, then $\mathcal{T}/{=_{\mathcal{T}}}$ has top $[\top]$. In this structure $A \cap B$ is $\inf\{A, B\}$, the greatest lower bound of A and B. If $\mathcal{T}$ is also compatible, then $\rightarrow$ can be properly defined on the equivalence classes. This will be done in Section 15.3.

Specific intersection type theories

Now we will construct several, in total thirteen, type theories that will play an important role in later chapters, by introducing the following axiom schemes, rule schemes and axioms. Only two of them are non-compatible, so we obtain eleven type structures.

In the following φ, ω and $\top$ are distinct atoms differing from those in $\mathbb{A}_\infty$.

15.1.6. Notation. We introduce names for axiom(scheme)s and rule(scheme)s in Figure 15.1. Using these names a list of well-studied type structures can be specified in Figure 15.2 as the set of judgements axiomatized by mentioned rule(scheme)s and axiom(scheme)s.

Axioms	
(ω_{Scott})	$(\top{\to}\omega) = \omega$
(ω_{Park})	$(\omega{\to}\omega) = \omega$
$(\omega\varphi)$	$\omega \leq \varphi$
$(\varphi{\to}\omega)$	$(\varphi{\to}\omega) = \omega$
$(\omega{\to}\varphi)$	$(\omega{\to}\varphi) = \varphi$
(I)	$(\varphi{\to}\varphi) \cap (\omega{\to}\omega) = \varphi$
Axiom schemes	
$(\top)$	$A \leq \top$
$(\top{\to})$	$\top \leq (A{\to}\top)$
$(\top_{\text{lazy}})$	$(A{\to}B) \leq (\top{\to}\top)$
$({\to}\cap)$	$(A{\to}B) \cap (A{\to}C) \leq A{\to}B \cap C$
$({\to}\cap^=)$	$(A{\to}B) \cap (A{\to}C) = A{\to}B \cap C$
Rule schemes	
$({\to})$	$\dfrac{A' \leq A \quad B \leq B'}{(A{\to}B) \leq (A'{\to}B')}$
$({\to}^=)$	$\dfrac{A' = A \quad B = B'}{(A{\to}B) = (A'{\to}B')}$

Figure 15.1: Possible Axioms and Rules concerning $\leq$.

15.1.7. Definition. In Figure 15.2 a collection of TTs is defined. For each name $\mathcal{T}$ a set of atoms $\mathbb{A}^{\mathcal{T}}$ and a set of rules and axiom(scheme)s are given. The type theory $\mathcal{T}$ is the smallest set of judgements of the form $A \leq B$ with $A, B \in \mathbb{T}^{\mathcal{T}} = \mathbb{T}_\cap^{\mathbb{A}^{\mathcal{T}}}$ which is closed under the axiom(scheme)s and the rule(scheme)s of Definition 15.1.1 and the corresponding ones in Figure 15.2.

15.1.8. Remark. (i) Note that CDS and CD are non-compatible, while the other eleven are compatible.

$\mathcal{T}$	$\mathbb{A}^{\mathcal{T}}$	Rules	Axiom Schemes	Axioms
Scott	$\{\top, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	(ω_{Scott})
Park	$\{\top, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	(ω_{Park})
CDZ	$\{\top, \varphi, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	$(\omega\varphi), (\varphi\rightarrow\omega), (\omega\rightarrow\varphi)$
HR	$\{\top, \varphi, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	$(\omega\varphi), (\varphi\rightarrow\omega), (I)$
DHM	$\{\top, \varphi, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	$(\omega\varphi), (\omega\rightarrow\varphi), (\omega_{\text{Scott}})$
BCD	$\mathbb{A}_\infty^\top$	$(\rightarrow)$	$(\rightarrow\cap), (\top), (\top\rightarrow)$	
AO	$\{\top\}$	$(\rightarrow)$	$(\rightarrow\cap), (\top)$	$(\top_{\text{lazy}})$
Plotkin	$\{\top, \omega\}$	$(\rightarrow^=)$	$(\top)$	–
Engeler	$\mathbb{A}_\infty^\top$	$(\rightarrow^=)$	$(\rightarrow\cap^=), (\top), (\top\rightarrow)$	–
CDS	$\mathbb{A}_\infty^\top$	–	$(\top)$	–
HL	$\{\varphi, \omega\}$	$(\rightarrow)$	$(\rightarrow\cap)$	$(\omega\varphi), (\omega\rightarrow\varphi), (\varphi\rightarrow\omega)$
CDV	$\mathbb{A}_\infty$	$(\rightarrow)$	$(\rightarrow\cap)$	–
CD	$\mathbb{A}_\infty$	–	–	–

Figure 15.2: Various type theories

(ii) The first ten type theories of Figure 15.2 belong clearly to $\text{TT}^\top$. In Lemma 15.1.14(i) we will see that also $\text{HL} \in \text{TT}^\top$ with φ as top. Instead CDS and CD do not belong to $\text{TT}^\top$, as shown in Lemma 15.1.14(ii) and (iii).

In this list the given order is logical, rather than historical, and some of the references define the models directly, others deal with the corresponding filter models (see Sections 17 and 18): Scott [1972], Park [1976], Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992], Dezani-Ciancaglini et al. [2005], Barendregt et al. [1983], Abramsky and Ong [1993], Plotkin [1993], Engeler [1981], Coppo et al. [1979], Honsell and Lenisa [1999], Coppo et al. [1981], Coppo and Dezani-Ciancaglini [1980]. These theories are denoted by names (respectively acronymes) of the author(s) who have first considered the λ-model induced by such a theory.

The expressive power of intersection types is remarkable. This will become apparent when we will use them as a tool for characterizing properties of λ-terms (see Sections 19.2 and 18.3), and for describing different λ-models (see Section 18). Much of this expressive power comes from the fact that they are endowed with a *preorder relation*, $\leq$, which induces, on the set of types modulo $=$, the structure of a meet semi-lattice with respect to $\cap$. This appears natural when we think of types as subsets of a domain of discourse D, which is endowed with a (partial) application $\cdot : D \times D \rightarrow D$, and interpret $\cap$ as set-theoretic intersection,

$\leq$ as set inclusion, and give $\rightarrow$ the *realizability interpretation.*

$$\begin{array}{rcl} [\![A]\!] & \subseteq & D \\ A \leq B & \Leftrightarrow & [\![A]\!] \subseteq [\![B]\!] \\ [\![A \cap B]\!] & = & [\![A]\!] \cap [\![B]\!] \\ [\![A \rightarrow B]\!] & = & [\![A]\!] \rightarrow [\![B]\!] = \{d \in D \mid d \cdot [\![A]\!] \subseteq [\![B]\!]\}. \end{array}$$

This semantics, due to Scott, will be studied in Section 19.1.

The type $\top \rightarrow \top$ is the set of functions which applied to an arbitrary element return again an arbitrary element. In that case axiom scheme $(\top \rightarrow)$ expresses the fact that all the objects in our domain of discourse are total functions, i.e. that $\top$ is equal to $A \rightarrow \top$, hence $A \rightarrow \top = B \rightarrow \top$ for all A, B (Barendregt et al. [1983]). If now we want to capture only those terms which truly represent functions, as we do for example in the lazy λ-calculus, we cannot assume axiom $(\top \rightarrow)$. One still may postulate the weaker property $(\top_{\text{lazy}})$ to make all functions total (Abramsky and Ong [1993]). It simply says that an element which is a function, because it maps A into B, maps also the whole universe into itself.

In Figure 15.3 below consider $\vdash_{\cap\top}^{\mathcal{T}}$ for the ten type theories above the horizontal line and $\vdash_{\cap}^{\mathcal{T}}$ for the other three. Define $\mathcal{T}_1 \subseteq \mathcal{T}_2$ as

$$\forall \Gamma, M, A.\ [\Gamma \vdash^{\mathcal{T}_1} M : A \ \Rightarrow\ \Gamma \vdash^{\mathcal{T}_2} M : A].$$

If this is the case we have connected $\mathcal{T}_1$ with an edge towards the higher positioned $\mathcal{T}_2$. In Exercise 16.3.21 we will show that the edges denote strict inclusions.

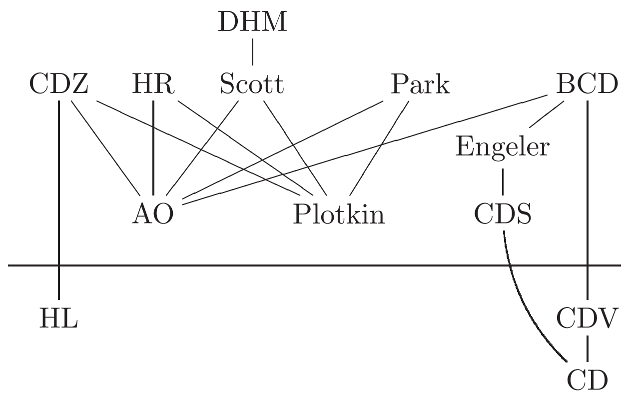

Figure 15.3: Inclusion among some intersection type theories.

The intended interpretation of arrow types also motivates axiom $(\rightarrow\cap)$, which implies that if a function maps A into B, and the same function maps also A into C, then, actually, it maps the whole A into the intersection between B and C (i.e. into $B \cap C$), see Barendregt et al. [1983].

Rule $(\rightarrow)$ is again very natural in view of the set-theoretic interpretation. It implies that the arrow constructor is contravariant in the first argument and covariant in the second one. It is clear that if a function maps A into B, and we take a subset A' of A and a superset B' of B, then this function will map also A' into B', see Barendregt et al. [1983].

The rule $(\to\cap^{=})$ is similar to the rule $(\to\cap)$. It capture properties of the graph models for the untyped lambda calculus, see Plotkin [1975] and Engeler [1981], as we shall discuss in Section 19.3.

In order to capture aspects of the λI-calculus we introduce TTs without an explicit mention of a top.

The remaining axioms express peculiar properties of D_∞-like inverse limit models, see Barendregt et al. [1983], Coppo et al. [1984], Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992], Honsell and Lenisa [1993], Alessi, Dezani-Ciancaglini and Honsell [2004]. We shall discuss them in more detail in Section 19.3.

Some classes of type theories

Now we will consider some classes of TT. In order to do this, we list the relevant defining properties.

15.1.9. Definition. We define special subclasses of TT.

Class	Defining axiom(-scheme)(s) or rule
graph	$(\to^{=}), (\to\cap^{=}), (\top)$
lazy	$(\to), (\to\cap), (\top), (\top_{\text{lazy}})$
natural	$(\to), (\to\cap), (\top), (\top\to)$
proper	$(\to), (\to\cap)$

15.1.10. Notation. The sets of graph, lazy, natural and proper type theories are denoted by respectively $\text{GTT}^\top$, $\text{LTT}^\top$ $\text{NTT}^\top$ and PTT.

15.1.11. Remark. The type theories of Figure 15.2 are classified as follows.

non compatible	CD, CDS
$\text{GTT}^\top$	Plotkin, Engeler
$\text{LTT}^\top$	AO
$\text{NTT}^\top$	Scott, Park, CDZ, HR, DHM, BCD
PTT	CDV, HL

15.1.12. Remark. One has $\text{NTT}^\top \subseteq \text{LTT}^\top \subseteq \text{GTT}^\top \subseteq \text{TT}$ and $\text{LTT}^\top \subseteq \text{PTT} \subseteq \text{TT}$. These inclusions are sharp.

Some properties about specific TTs

Results about proper type theories

15.1.13. PROPOSITION. *Let $\mathcal{T}$ be a proper type theory. Then we have*

(i) $(A{\to}B)\cap(A'{\to}B') \leq (A\cap A'){\to}(B\cap B')$;

(ii) $(A_1{\to}B_1)\cap\ldots\cap(A_n{\to}B_n) \leq (A_1\cap\ldots\cap A_n){\to}(B_1\cap\ldots\cap B_n)$;

(iii) $(A{\to}B_1)\cap\ldots\cap(A{\to}B_n) = A{\to}(B_1\cap\ldots B_n)$.

PROOF. (i) $$\begin{array}{rcl}(A{\to}B)\cap(A'{\to}B') & \leq & ((A\cap A'){\to}B)\cap((A\cap A'){\to}B') \\ & \leq & (A\cap A'){\to}(B\cap B'),\end{array}$$
by respectively $(\to)$ and $(\to\cap)$.

(ii) Similarly (i.e. by induction on $n{>}1$, using (i) for the induction step).

(iii) By (ii) one has $(A{\to}B_1)\cap\ldots\cap(A{\to}B_n) \leq A{\to}(B_1\cap\ldots B_n)$. For $\geq$ use $(\to)$ to show that $A{\to}(B_1\cap\ldots\cap B_n) \leq (A{\to}B_i)$, for all i. ■

It follows that the mentioned equality and inequalities hold for Scott, Park, CDZ, HR, DHM, BCD, AO, HL and CDV.

Results about the type theories of Figure 15.2

15.1.14. LEMMA. (i) *φ is the top and ω the bottom element in* HL.

(ii) CDV *has no top element.*

(iii) CD *has no top element.*

PROOF. (i) By induction on the generation of $\mathbb{T}^{\mathrm{HL}}$ one shows that $\omega \leq A \leq \varphi$ for all $A\in\mathbb{T}^{\mathrm{HL}}$.

(ii) If α is a fixed atom and

$$\mathcal{B}_\alpha := \alpha \mid \mathcal{B}_\alpha \cap \mathcal{B}_\alpha$$

and $A\in\mathcal{B}_\alpha$, then one can show by induction on the generation of $\leq_{\mathrm{CDV}}$ that $A \leq_{\mathrm{CDV}} B \;\Rightarrow\; A\in\mathcal{B}_\alpha$. Hence if $\alpha \leq_{\mathrm{CDV}} B$, then $B\in\mathcal{B}_\alpha$. Since $\mathcal{B}_{\alpha_1}$ and $\mathcal{B}_{\alpha_2}$ are disjoint when α_1 and α_2 are two different atoms, we conclude that CDV has no top element.

(iii) Similar to (ii). ■

15.1.15. REMARK. By the above lemma φ turns out to be the top element in HL. But we will not use this and therefore denote it by the name φ and not $\top$.

In the following lemmas 15.1.16-15.1.20 we study the positions of the atoms ω, and φ in the compatible TTs introduced in Figure 15.2. The principal result is that $\omega < \varphi$ in HL and, as far as applicable,

$$\omega < \varphi < \top,$$

in the theories Scott, Park, CDZ, HR, DHM and Plotkin.

15.1.16. LEMMA. *Let $\mathcal{T} \in \{\text{Scott}, \text{Park}, \text{CDZ}, \text{HR}, \text{DHM}, \text{BCD}, \text{Engeler}\}$ be as defined in Figure 15.2. Define inductively the following collection of types*

$$\boxed{\mathcal{B} \quad := \quad \top \mid \mathbb{T}^{\mathcal{T}} {\to} \mathcal{B} \mid \mathcal{B} \cap \mathcal{B}}$$

Then $\mathcal{B} = \{A \in \mathbb{T}^{\mathcal{T}} \mid A =_{\mathcal{T}} \top\}$.

PROOF. By induction on the generation of $A \leq_{\mathcal{T}} B$ one proves that $\mathcal{B}$ is closed upwards. This gives $\top \leq A \;\Rightarrow\; A \in \mathcal{B}$.

By induction on the definition of $\mathcal{B}$ one shows, using $(\top{\to})$ and $({\to})$, that $A \in \mathcal{B} \;\Rightarrow\; \top \leq A$.

Therefore
$$A =_{\mathcal{T}} \top \;\Leftrightarrow\; \top \leq A \;\Leftrightarrow\; A \in \mathcal{B}. \blacksquare$$

15.1.17. LEMMA. *For $\mathcal{T} \in \{\text{AO}, \text{Plotkin}\}$ define inductively*

$$\boxed{\mathcal{B} \quad := \quad \top \mid \mathcal{B} \cap \mathcal{B}}$$

Then $\mathcal{B} = \{A \in \mathbb{T}^{\mathcal{T}} \mid A =_{\mathcal{T}} \top\}$, hence $\top{\to}\top \neq_{\mathcal{T}} \top$.

PROOF. Similar to the proof of 15.1.14, but easier. ■

15.1.18. LEMMA. *For $\mathcal{T} \in \{\text{CDZ}, \text{HR}, \text{DHM}\}$ define by mutual induction*

$$\begin{array}{rcl} \mathcal{B} & = & \varphi \mid \top \mid \mathbb{T}^{\mathcal{T}} {\to} \mathcal{B} \mid \mathcal{H} {\to} \mathbb{T}^{\mathcal{T}} \mid \mathcal{B} \cap \mathcal{B} \\ \mathcal{H} & = & \omega \mid \mathcal{B} {\to} \mathcal{H} \mid \mathcal{H} \cap \mathbb{T}^{\mathcal{T}} \mid \mathbb{T}^{\mathcal{T}} \cap \mathcal{H} \end{array}$$

Then

$$\begin{array}{rcl} \varphi \leq B & \Rightarrow & B \in \mathcal{B}, \\ A \leq \omega & \Rightarrow & A \in \mathcal{H}. \end{array}$$

PROOF. By induction on $\leq_{\mathcal{T}}$ one shows

$$A \leq B \;\Rightarrow\; (A \in \mathcal{B} \;\Rightarrow\; B \in \mathcal{B}) \;\Rightarrow\; (B \in \mathcal{H} \;\Rightarrow\; A \in \mathcal{H}).$$

From this the assertion follows immediately. ■

15.1.19. LEMMA. *We work with the theory* HL.

(i) *Define by mutual induction*

$$\begin{array}{rcl} \mathcal{B} & = & \varphi \mid \mathcal{H} {\to} \mathcal{B} \mid \mathcal{B} \cap \mathcal{B} \\ \mathcal{H} & = & \omega \mid \mathcal{B} {\to} \mathcal{H} \mid \mathcal{H} \cap \mathbb{T} \mid \mathbb{T} \cap \mathcal{H} \end{array}$$

Then

$$\begin{array}{rcl} \mathcal{B} & = & \{A \in \mathbb{T}^{\text{HL}} \mid A =_{\text{HL}} \varphi\}; \\ \mathcal{H} & = & \{A \in \mathbb{T}^{\text{HL}} \mid A =_{\text{HL}} \omega\}. \end{array}$$

(ii) *$\omega \neq_{\text{HL}} \varphi$ and hence $\omega <_{\text{HL}} \varphi$.*

PROOF. (i) By induction on $\leq_{\mathcal{T}}$ one shows

$$A \leq B \;\Rightarrow\; (A \in \mathcal{B} \;\Rightarrow\; B \in \mathcal{B})\ \&\ (B \in \mathcal{H} \;\Rightarrow\; A \in \mathcal{H}).$$

This gives

$$(\varphi \leq B \;\Rightarrow\; B \in \mathcal{B})\ \&\ (A \leq \omega \;\Rightarrow\; A \in \mathcal{H}).$$

By simultaneous induction on the generation of $\mathcal{B}$ and $\mathcal{H}$ one shows, using that ω is the bottom element of HL, by Lemma 15.1.14(i),

$$(B \in \mathcal{B} \;\Rightarrow\; B = \varphi)\ \&\ (A \in \mathcal{H} \;\Rightarrow\; A = \omega).$$

Now the assertion follows immediately.

(ii) By (i). ■

15.1.20. PROPOSITION. *In* HL *we have* $\omega < \varphi$ *and as far as applicable we have for the other systems of Figure 15.2*

$$\omega < \varphi < \top.$$

More precisely,

(i) $\omega \leq \varphi$ *and* $\omega \neq \varphi$ *in* HL.

In all other systems

(ii) $\omega \leq \varphi$, $\omega \leq \top$, $\varphi \leq \top$;

(iii) $\omega \neq \varphi$, $\omega \neq \top$, $\varphi \neq \top$.

PROOF. (i) By $(\omega\varphi)$ and Lemma 15.1.19.

(ii) By $(\omega\varphi)$ and $(\top)$.

(iii) By Lemmas 15.1.16-15.1.18. ■

15.2. Type assignment

Assignment of types from type theories

In this subsection we define for a $\mathcal{T}$ in TT a type assignment system $\lambda_\cap^{\mathcal{T}}$, that assigns to untyped lambda terms a (possibly empty set of) types in $\mathbb{T}^{\mathcal{T}}$. For a $\mathcal{T}$ in TT$^\top$ we also define a type assignment system $\lambda_{\cap\top}^{\mathcal{T}}$.

15.2.1. DEFINITION. (i) A $\mathcal{T}$-*statement* is of the form $M : A$ with the *subject* an untyped lambda term $M \in \Lambda$ and the *predicate* a type $A \in \mathbb{T}^{\mathcal{T}}$.

(ii) A $\mathcal{T}$-*declaration* is a $\mathcal{T}$-statement of the form $x : A$.

(iii) A $\mathcal{T}$-*basis* Γ is a finite set of $\mathcal{T}$-declarations, with all variables distinct.

(iv) A $\mathcal{T}$-*assertion* is of the form $\Gamma \vdash M : A$, where $M : A$ is a $\mathcal{T}$-statement and Γ is a $\mathcal{T}$-basis.

15.2.2. DEFINITION. (i) The *(basic) type assignment system* $\lambda_\cap^{\mathcal{T}}$ derives $\mathcal{T}$-assertions by the following axioms and rules.

(Ax)	$\Gamma \vdash x{:}A$	if $(x{:}A \in \Gamma)$
$(\to\mathrm{I})$	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A{\to}B}$	
$(\to\mathrm{E})$	$\dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$	
$(\cap\mathrm{I})$	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$	
$(\leq)$	$\dfrac{\Gamma \vdash M : A \quad A \leq_{\mathcal{T}} B}{\Gamma \vdash M : B}$	

Figure 15.4: Basic type assignment system $\lambda_{\cap}^{\mathcal{T}}$.

(ii) If $\mathcal{T}$ has a top element $\top$, then the $\top$-*type assignment system* $\lambda_{\cap^\top}^{\mathcal{T}}$ is defined by adding the extra axiom to the basic system

($\top$-universal)	$\Gamma \vdash M : \top$

Figure 15.5: The extra axiom for the top assignment system $\lambda_{\cap^\top}^{\mathcal{T}}$

15.2.3. Notation. (i) We write $\Gamma \vdash_{\cap^\top}^{\mathcal{T}} M : A$ or $\Gamma \vdash_{\cap}^{\mathcal{T}} M : A$ if $\Gamma \vdash M : A$ is derivable in $\lambda_{\cap^\top}^{\mathcal{T}}$ or $\lambda_{\cap}^{\mathcal{T}}$ respectively.

(ii) The assertion $\vdash_{\cap^\top}^{\mathcal{T}}$ may also be written as $\vdash^{\mathcal{T}}$, $\vdash_{\cap^\top}$ or simply $\vdash$ if by the context there is little danger of confusion. Similarly, $\vdash_{\cap}^{\mathcal{T}}$ may be written as $\vdash^{\mathcal{T}}$, $\vdash_{\cap}$ or $\vdash$.

(iii) $\lambda_{\cap^{(\top)}}^{\mathcal{T}}$ may be denoted by $\lambda_{\cap^{(\top)}}$.

15.2.4. Example. Let $\mathcal{T} \in \mathrm{TT}^{\top}$ with $A, B \in \mathbb{T}^{\mathcal{T}}$. Write $W \equiv (\lambda x.xx)$.

(i) $\vdash_{\cap}^{\mathcal{T}} W : A \cap (A{\to}B){\to}B$.

$\vdash_{\cap^\top}^{\mathcal{T}} WW : \top$, but WW does not have a type in $\lambda_{\cap}^{\mathcal{T}}$.

(ii) Let $M \equiv \mathsf{KI}(WW)$. Then $\vdash M : (A{\to}A)$ in $\lambda_{\cap^\top}^{\mathcal{T}}$.

(iii) (van Bakel) Let $M \equiv \lambda yz.\mathsf{K}z(yz)$ and $N \equiv \lambda yz.z$. Then $M \twoheadrightarrow_{\boldsymbol{\beta}} N$. We have $\vdash_{\cap}^{\mathcal{T}} N : B{\to}A{\to}A$, $\vdash_{\cap^\top}^{\mathcal{T}} M : B{\to}A{\to}A$, but $\not\vdash_{\cap}^{\mathcal{T}} M : B{\to}A{\to}A$.

(iv) $\not\vdash_{\cap}^{\mathrm{CD}} \mathsf{I} : ((A \cap B){\to}C){\to}((B \cap A){\to}C)$.

In general the type assignment systems $\lambda_{\cap^\top}^{\mathcal{T}}$ will be used for the the $\lambda\mathsf{K}$-calculus and $\lambda_{\cap}^{\mathcal{T}}$ for the $\lambda\mathsf{I}$-calculus.

15.2.5. Definition. Define the rules $(\cap\mathrm{E})$

$$\frac{\Gamma \vdash M : (A \cap B)}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : (A \cap B)}{\Gamma \vdash M : B}$$

Notice that these rules are derived in $\lambda_{\cap}^{\mathcal{T}}$, $\lambda_{\cap^\top}^{\mathcal{T}}$ for all $\mathcal{T}$.

15.2.6. Lemma. *In* $\lambda_{\cap}^{\mathcal{T}}$ *one has the following.*

(i) $\Gamma \vdash M : A \;\Rightarrow\; \mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma)$.
(ii) $\Gamma \vdash M : A \;\Rightarrow\; (\Gamma \upharpoonright \mathrm{FV}(M)) \vdash M : A$.
(iii) *If in $\mathcal{T}$ with top $\top$ one has $\top = \top{\to}\top$, then*

$$\mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma) \;\Rightarrow\; \Gamma \vdash M : \top.$$

Proof. (i), (ii) By induction on the derivation.
(iii) By induction on M. ■

Notice that $\Gamma \vdash M : A \Rightarrow \mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma)$ does not hold in $\lambda_{\cap\top}^{\mathcal{T}}$, since by axiom ($\top$ universal) we have $\vdash^{\mathcal{T}} M : \top$ for all $\mathcal{T}$ and all M.

15.2.7. Remark. For the type theories of Figure 15.2 with $\top$ we have defined the type assignment systems $\lambda_\cap^{\mathcal{T}}$. For those system having a top, there is also the type assignment system $\lambda_{\cap\top}^{\mathcal{T}}$. We will use for the type theories in Figure 15.2 only one of the two possibilities. For the first ten systems, i.e. Scott, Park, CDZ, HR, DHM, BCD, AO, Plotkin, Engeler and CDS, we only consider $\lambda_{\cap\top}^{\mathcal{T}}$. For the other 3 systems, i.e. HL, CDV and CD, we will only consider $\lambda_\cap^{\mathcal{T}}$. In fact by Lemma 15.1.14(ii) and (iii) we know that CDV and CD have no top at all. The system HL has a top, but we will not use it, as we do not know interesting properties of $\lambda_{\cap\top}^{\mathrm{HL}}$. So, for example, $\vdash^{\mathrm{Scott}}$ will be always $\vdash_{\cap\top}^{\mathrm{Scott}}$, whereas $\vdash^{\mathrm{HL}}$ will be always $\vdash_\cap^{\mathrm{HL}}$. The reader will be reminded of this. We do not know wether there exist TTs where the interplay of $\lambda_\cap^{\mathcal{T}}$ and $\lambda_{\cap\top}^{\mathcal{T}}$ yields results of interest.

Admissible rules

15.2.8. Proposition. *The following rules are admissible in $\lambda_{\cap(\top)}^{\mathcal{T}}$.*

(*weakening*)	$\dfrac{\Gamma \vdash M : A \quad x \notin \Gamma}{\Gamma, x{:}B \vdash M : A}$;
(*strengthening*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad x \notin FV(M)}{\Gamma \vdash M : A}$;
(*cut*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M[x := N]) : A}$;
($\leq$-L)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad C \leq_{\mathcal{T}} B}{\Gamma, x{:}C \vdash M : A}$;
($\to$L)	$\dfrac{\Gamma, y{:}B \vdash M : A \quad \Gamma \vdash N : C \quad x \notin \Gamma}{\Gamma, x{:}(C{\to}B) \vdash (M[y := xN]) : A}$;
($\cap$L)	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma, x{:}(A \cap C) \vdash M : B}$.

Figure 15.6: Various admissible rules.

Proof. By induction on the structure of derivations. ■

Proofs later on in Part III will freely use the rules of the above proposition.

As we remarked earlier, there are various equivalent alternative presentations of intersection type assignment systems. We have chosen a natural deduction presentation, where $\mathcal{T}$-bases are additive. We could have taken, just as well, a sequent style presentation and replace rule ($\to$E) with the three rules ($\to$L), ($\cap$L) and (cut) occuring in Proposition 15.2.8, see Barbanera et al. [1995], Barendregt and Ghilezan [n.d.]. Next to this we could have formulated the rules so that $\mathcal{T}$-bases "multiply". Notice that because of the presence of the type constructor $\cap$, a special notion of *multiplication of $\mathcal{T}$-bases* can be given.

15.2.9. Definition (Multiplication of $\mathcal{T}$-bases).

$$\begin{array}{rcl} \Gamma \uplus \Gamma' & = & \{x{:}\, A \cap B \mid x{:}\, A \in \Gamma \text{ and } x{:}\, B \in \Gamma'\} \\ & \cup & \{x{:}\, A \mid x{:}\, A \in \Gamma \text{ and } x \notin \Gamma'\} \\ & \cup & \{x{:}\, B \mid x{:}\, B \in \Gamma' \text{ and } x \notin \Gamma\}. \blacksquare \end{array}$$

Accordingly we define:

$$\Gamma \Subset \Gamma' \;\Leftrightarrow\; \exists \Gamma''.\, \Gamma \uplus \Gamma'' = \Gamma'.$$

For example, $\{x{:}A, y{:}B\} \uplus \{x{:}C, z{:}D\} = \{x{:}A \cap C, y{:}B, z{:}D\}$.

15.2.10. Proposition. *The following rules are admissible in all $\lambda^{\mathcal{T}}_{\cap(\top)}$.*

(*multiple weakening*)	$\dfrac{\Gamma_1 \vdash M : A}{\Gamma_1 \uplus \Gamma_2 \vdash M : A}$
(*relevant* $\to$E)	$\dfrac{\Gamma_1 \vdash M : A \to B \quad \Gamma_2 \vdash N : A}{\Gamma_1 \uplus \Gamma_2 \vdash MN : B}$
(*relevant* $\cap$I)	$\dfrac{\Gamma_1 \vdash M : A \quad \Gamma_2 \vdash M : B}{\Gamma_1 \uplus \Gamma_2 \vdash M : A \cap B}$

Proof. By induction on derivations. $\blacksquare$

In Exercise 16.3.17, it will be shown that we can replace rule ($\leq$) with other more perspicuous rules. This is possible as soon as we will have proved appropriate "inversion" theorems for $\lambda^{\mathcal{T}}_{\cap(\top)}$. For some very special theories, one can even omit altogether rule ($\leq$), provided the remaining rules are reformulated "multiplicatively" with respect to $\mathcal{T}$-bases, see e.g. Di Gianantonio and Honsell [1993]. We shall not follow up this line of investigation.

In $\lambda^{\mathcal{T}}_{\cap(\top)}$, assumptions are allowed to appear in the basis without any restriction. Alternatively, we might introduce a *relevant* intersection type assignment system, where only "minimal-base" judgements are derivable, (see Honsell and Ronchi Della Rocca [1992]). Rules like (*relevant* $\to$E) and (*relevant* $\cap$I), which exploit the above notion of multiplication of bases, are essential for this purpose. Relevant systems are necessary, for example, for giving finitary logical descriptions of qualitative domains as defined in Girard et al. [1989]. We will not follow up this line of research either. See Honsell and Ronchi Della Rocca [1992].

Special type assignment for call-by-value λ-calculus

We will study later the type theory EHR with $\mathbb{A}^{\text{EHR}} = \{\nu\}$ and the extra rule $(\rightarrow)$ and axioms $(\rightarrow\cap)$ and

$$A\rightarrow B \leq \nu.$$

The type assignment system $\lambda_{\cap\nu}^{\text{EHR}}$ is defined by the axiom and rules of $\lambda_\cap^{\mathcal{T}}$ in Figure 15.4 with the extra axiom

$$\boxed{(\nu \text{ universal}) \qquad \Gamma \vdash (\lambda x.M) : \nu.}$$

The type theory EHR has a top, namely ν, so one could consider it as an element of $\text{TT}^\top$. This will not be done. Axiom (ν-universal) is different from ($\top$-universal) in Definition 15.2.2. This type assignment system has one particular application and will be studied in some exercises.

15.3. Type structures

Intersection type structures

Remember that a type algebra $\mathcal{A}$, see Definition **??**, is of the form $\mathcal{A} = \langle|\mathcal{A}|, \rightarrow\rangle$, i.e. just an arbitrary set $|\mathcal{A}|$ with a binary operation $\rightarrow$ on it.

15.3.1. Definition. (i) A *meet semi-lattice* is a structure

$$\mathcal{M} = \langle|\mathcal{M}|, \leq, \cap\rangle,$$

such that $\mathcal{M} = \langle|\mathcal{M}|, \leq, \cap\rangle$ is a partial order, for all $A, B \in |\mathcal{M}|$ the element $A \cap B$ (meet) is the greatest lower bound of A and B. MSL is the set of meet semi-lattices.

(ii) A *top meet semi-lattice* is a similar structure

$$\mathcal{M} = \langle|\mathcal{M}|, \leq, \cap, \top\rangle,$$

such that $\mathcal{M} = \langle|\mathcal{M}|, \leq, \cap\rangle$ is a MSL and $\top$ is the (unique) top of $\mathcal{M}$. $\text{MSL}^\top$ is the set of top meet semi-lattices.

15.3.2. Definition. (i) An *(intersection) type structure* is a type algebra with the additional structure of a meet semi-lattice

$$\mathcal{S} = \langle|\mathcal{S}|, \rightarrow, \leq, \cap\rangle.$$

TS is the set of type structures. The relation $\leq$ and the operation $\rightarrow$ have a priori no relation with each other, but in special structures this will be the case.

(ii) A *top type structure* is a type algebra that is also a top meet semi-lattice

$$\mathcal{S} = \langle|\mathcal{S}|, \rightarrow, \leq, \cap, \top\rangle.$$

$\text{TS}^\top$ is the set of top type structures.

NOTATION. (i) As 'intersection' is everywhere in this Part III, we will omit this word and only speak about a *type structure*.

(ii) *Par abus de language* we also use $A, B, C, \ldots$ to denote arbitrary elements of type structures and we write $A \in \mathcal{S}$ for $A \in |\mathcal{S}|$.

If $\mathcal{T}$ is a type theory that is not compatible, like CD and CDS, then $\rightarrow$ cannot be defined on the equivalence classes. But if $\mathcal{T}$ is compatible, then one can work on the equivalence classes and obtain a type structure in which $\leq$ is a partial order.

15.3.3. PROPOSITION. *Let $\mathcal{T}$ be a compatible type theory. Then $\mathcal{T}$ induces a type structure $\mathcal{T}/=$ defined as follows.*

$$\langle \mathbb{T}^{\mathcal{T}}/{=_{\mathcal{T}}}, \rightarrow, \leq, \cap \rangle,$$

by defining on the $=_{\mathcal{T}}$-equivalence classes

$$\begin{array}{rcl} [A]\rightarrow[B] & = & [A\rightarrow B]^2; \\ [A]\cap[B] & = & [A\cap B]; \\ [A]\leq[B] & \Leftrightarrow & A \leq_{\mathcal{T}} B. \end{array}$$

If moreover $\mathcal{T}$ has a top $\top$, then $\mathcal{T}/=$ is a top type structure with $[\top]$ as top.

PROOF. Here A, B, C range over $\mathbb{T}^{\mathcal{T}}$. Having realized this the rest is easy. Rule $(\rightarrow^{=})$ is needed to ensure that $\rightarrow$ is well-defined. ∎

The (top) type structure $\mathcal{T}/=$, with $\mathcal{T}$ a type theory, is called a *syntactical* (top) type structure. In Proposition 15.3.6 we show that every type structure is isomorphic to a syntactical one.

Although essentially equivalent, type structures and type theories differ in the following. In the theories the types are freely generated from a fixed set of atoms and inequality can be controlled somewhat by choosing the right axioms and rules (this will be exploited in Section 19.3). In type structures one has the antisymmetric law $A \leq B \leq A \;\Rightarrow\; A = B$, which is in line with the common theory of partial orders (this will be exploited in Chapter 17).

Now the notion of type assignment will also be defined for intersection type structures. These structures arise naturally coming from algebraic lattices that are used towards obtaining a semantics for untyped lambda calculus.

15.3.4. DEFINITION. (i) Now let $\mathcal{S} \in \mathrm{TS}$. The notion of a $\mathcal{S}$-statement $M : A$, a $\mathcal{S}$-declaration $x : A$, a $\mathcal{S}$-basis and a $\mathcal{S}$-assertion $\Gamma \vdash M : A$ is as in Definition 15.2.1, now for $A \in \mathcal{S}$ an element of the type structure $\mathcal{S}$.

(ii) The notion $\Gamma \vdash^{\mathcal{S}}_{\cap} M : A$ is defined by the same set of axioms and rules as in Figure 15.4 where now $\leq_{\mathcal{S}}$ is the inequality of the structure $\mathcal{S}$. The assignment system $\lambda^{\mathcal{S}}_{\cap\top}$ has ($\top$-universal) as extra axiom.

2 Here we misuse notation in a suggestive way, by using the same notation $\rightarrow$ for equivalence classes as for types.

The following result shows that for syntactic type structures type assignment is essentially the same as the one coming from the corresponding lambda theory.

15.3.5. Proposition. *Let $\mathcal{T} \in TT^{(\top)}$ and let $[\mathcal{T}] = \langle \mathcal{T}/=_{\mathcal{T}}, \leq, \cap, \rightarrow (,\top)\rangle$ its corresponding (top) type structure. For a type $A \in \mathcal{T}$ write its equivalence class as $[A] \in [\mathcal{T}]$. For $\Gamma = \{x_1 : B_1, \ldots, x_n : B_n\}$ a $\mathcal{T}$-basis write $[\Gamma] = \{x_1 : [B_1], \ldots, x_n : [B_n]\}$, a $[\mathcal{T}]$-basis. Then*

$$\Gamma \vdash_{\cap(\top)}^{\mathcal{T}} M : A \;\Leftrightarrow\; [\Gamma] \vdash_{\cap(\top)}^{[\mathcal{T}]} M : [A].$$

Proof. ($\Rightarrow$) By induction on the derivation of $\Gamma \vdash^{\mathcal{T}} M : A$. ($\Leftarrow$) Show by induction on the derivation of $[\Gamma] \vdash^{[\mathcal{T}]} M : [A]$ that for all $A' \in [A]$ and $\Gamma' = \{x_1 : B'_1, \ldots, x_n : B'_n\}$, with $B'_i \in [B_i]$ for all $1 \leq i \leq n$, one has

$$\Gamma' \vdash^{\mathcal{T}} M : A'. \blacksquare$$

Using this result we could have defined type assignment first for type structures and then for compatible type theories via translation to the type assignment for its corresponding syntactical type structure, essentially by turning the previous result into a definition.

15.3.6. Proposition. *Every type structure is isomorphic to a syntactical one.*

Proof. For a type structure $\mathcal{S}$, define $\mathcal{T}_{\mathcal{S}}$ as follows. Take $\mathbb{A} = \{\underline{c} \mid c \in \mathcal{S}\}$. Define $\leq_{\mathcal{T}_{\mathcal{S}}}$ on $\mathbb{T} = \mathbb{T}_{\cap}^{\mathbb{A}}$ as follows. We make every element of $\mathbb{T}$ equal to an element of $\mathbb{A}$ by requiring

$$(\underline{a} \cap \underline{b}) =_{\mathcal{T}_{\mathcal{S}}} \underline{a \cap b}, \;\&\; (\underline{a} \rightarrow \underline{b}) =_{\mathcal{T}_{\mathcal{S}}} \underline{a \rightarrow b}.$$

This means of course $(\underline{a} \cap \underline{b}) \leq_{\mathcal{T}_{\mathcal{S}}} \underline{a \cap b}$, $(\underline{a} \cap \underline{b}) \geq_{\mathcal{T}_{\mathcal{S}}} \underline{a \cap b}$, etcetera. We moreover require

$$\frac{a \leq_{\mathcal{S}} b}{\underline{a} \leq_{\mathcal{T}_{\mathcal{S}}} \underline{b}}.$$

As a consequence $\underline{a} \leq_{\mathcal{T}_{\mathcal{S}}} \underline{\top}$ if $\mathcal{S}$ is a top type structure. The axioms and rules (refl), (trans), ($\rightarrow^{=}$), (incl$_L$), (incl$_R$) and (glb) also hold automatically. Then $\mathcal{S} \cong \mathcal{T}_{\mathcal{S}}/=$. This can be seen as follows. Define $f : \mathcal{S} \rightarrow \mathcal{T}_{\mathcal{S}}/=$ by $f(a) = [\underline{a}]$. For the inverse, first define $g : \mathbb{T}_{\cap}^{\mathbb{A}} \rightarrow \mathcal{S}$ by

$$\begin{aligned} g(\underline{c}) &= c; \\ g(A \rightarrow B) &= g(A) \rightarrow g(B); \\ g(A \cap B) &= g(A) \cap g(B). \end{aligned}$$

Then show $A \leq_{\mathcal{T}_{\mathcal{S}}} B \;\Rightarrow\; g(A) \leq g(B)$. Finally set $f^{-1}([A]) = g(A)$, which is well defined. It is easy to show that f, f^{-1} constitute an isomorphism. $\blacksquare$

15.3.7. Remark. Each of the eleven compatible type theories $\mathcal{T}$ in Figure 15.2 may be considered as the intersection type structure $\mathcal{T}/=$, also denoted as $\mathcal{T}$. For example Scott can be a name, a type theory or a type structure.

Categories of meet-semi lattices and type structures

For use in Chapter 17 we will introduce some categories related to given classes of type structures.

15.3.8. DEFINITION. (i) The category **MSL** has as objects at most countable meet semi-lattices and as morphisms maps $f : \mathcal{M} \rightarrow \mathcal{M}'$, preserving $\leq, \cap$:

$$\begin{array}{rcl} A \leq B & \Rightarrow & f(A) \leq' f(B); \\ f(A \cap B) & = & f(A) \cap' f(B). \end{array}$$

(ii) The category $\mathbf{MSL}^{\top}$ is as **MSL**, but based on top meet semi-lattices. So now also $f(\top) = \top'$ for morphisms.

The difference between **MSL** and $\mathbf{MSL}^{\top}$ is that, in the **MSL** case, the top element is either missing or not relevant (not preserved by morphisms).

15.3.9. DEFINITION. (i) The category **TS** has as objects the at most countable type structures and as morphisms maps $f : \mathcal{S} \rightarrow \mathcal{S}'$, preserving $\leq, \cap, \rightarrow$:

$$\begin{array}{rcl} A \leq B & \Rightarrow & f(A) \leq' f(B); \\ f(A \cap B) & = & f(A) \cap' f(B); \\ f(A \rightarrow B) & = & f(A) \rightarrow' f(B). \end{array}$$

(ii) The category $\mathbf{TS}^{\top}$ is as **TS**, but based on top type structures. Now also

$$f(\top) = \top'$$

for morphisms.

15.3.10. DEFINITION. We define four full subcategories of **TS** by specifying in each case the objects.

(i) $\mathbf{GTS}^{\top}$ with as objects the graph top type structures.
(ii) $\mathbf{LTS}^{\top}$ with as objects the lazy top type structures.
(iii) $\mathbf{NTS}^{\top}$ with as objects the natural top type structures.
(iv) **PTS** with as objects the proper type structures.

15.4. Filters

15.4.1. DEFINITION. (i) Let $\mathcal{T} \in \mathrm{TT}$ and $X \subseteq \mathbb{T}^{\mathcal{T}}$. Then X is a *filter* over $\mathcal{T}$ if the following hold.

(1) X is non-empty;

(2) $A \in X$ & $A \leq B \Rightarrow B \in X$;

(3) $A, B \in X \Rightarrow A \cap B \in X$.

(ii) Write $\mathcal{F}^{\mathcal{T}} = \{X \subseteq \mathcal{T} \mid X \text{ is a filter over } \mathcal{T}\}$.

We loosely say that filters are non-empty sets of types closed under $\leq$ and $\cap$.

15.4.2. DEFINITION. Let $\mathcal{T} \in \mathrm{TT}$.

(i) For $A \in \mathbb{T}^{\mathcal{T}}$ write $\uparrow A = \{B \in \mathbb{T}^{\mathcal{T}} \mid A \leq B\}$.

(ii) For a non-empty $X \subseteq \mathbb{T}^{\mathcal{T}}$ define $\uparrow X$ to be the least filter over $\mathbb{T}^{\mathcal{T}}$ containing X; it can be described explicitly by

$$\uparrow X = \{B \in \mathbb{T}^{\mathcal{T}} \mid \exists n \geq 1 \, \exists A_1, \ldots, A_n \in X. A_1 \cap \ldots \cap A_n \leq B\}.$$

15.4.3. Remark. $C \in \uparrow \{B_i \mid i \in \mathcal{I} \neq \emptyset\} \Leftrightarrow \exists I \subseteq_{\text{fin}} \mathcal{I}.[I \neq \emptyset \ \& \ \bigcap_{i \in I} B_i \leq C]$.

15.4.4. Proposition. *Let $\mathcal{T} \in \mathrm{TT}^{\top}$.*

(i) *$\mathcal{F}^{\mathcal{T}} = \langle \mathcal{F}^{\mathcal{T}}, \subseteq \rangle$ is a complete lattice, with for $\mathcal{X} \subseteq \mathcal{F}^{\mathcal{T}}$ the sup is*

$$\begin{aligned} \bigsqcup \mathcal{X} &= \uparrow(\cup \mathcal{X}), && \textit{if } X \neq \emptyset, \\ \bigsqcup \mathcal{X} &= \{\top\}, && \textit{else.} \end{aligned}$$

(ii) *For $A \in \mathbb{T}^{\mathcal{T}}$ one has $\uparrow A = \uparrow \{A\}$ and $\uparrow A \in \mathcal{F}^{\mathcal{T}}$.*

(iii) *For $A, B \in \mathbb{T}^{\mathcal{T}}$ one has $\uparrow A \sqcup \uparrow B = \uparrow(A \cap B)$.*

(iv) *For $X \in \mathcal{F}^{\mathcal{T}}$ one has*

$$\begin{aligned} X &= \bigsqcup \{\uparrow A \mid A \in X\} \\ &= \bigsqcup \{\uparrow A \mid \uparrow A \subseteq X\} \\ &= \bigcup \{\uparrow A \mid A \in X\} \\ &= \bigcup \{\uparrow A \mid \uparrow A \subseteq X\}. \end{aligned}$$

(v) *$\{\uparrow A \mid A \in \mathbb{T}^{\mathcal{T}}\}$ is the set of finite elements of $\mathcal{F}^{\mathcal{T}}$.*

Proof. Easy. ■

15.4.5. Definition. Let $\mathcal{T} \in \mathrm{TT}$. Then $\mathcal{F}_s^{\mathcal{T}} = \mathcal{F}^{\mathcal{T}} \cup \{\emptyset\}$ is the extension of $\mathcal{F}^{\mathcal{T}}$ with the emptyset.

15.4.6. Proposition. *Let $\mathcal{T} \in \mathrm{TT}$.*

(i) *$\mathcal{F}_s^{\mathcal{T}} = \langle \mathcal{F}_s^{\mathcal{T}}, \subseteq \rangle$ is a complete lattice, with for $\mathcal{X} \subseteq \mathcal{F}_s^{\mathcal{S}}$ the sup is*

$$\bigsqcup \mathcal{X} = \begin{cases} \emptyset, & \textit{if } \mathcal{X} = \emptyset \textit{ or } \mathcal{X} = \{\emptyset\}, \\ \uparrow(\cup \mathcal{X}), & \textit{else .} \end{cases}$$

(ii) *For $A \in \mathbb{T}^{\mathcal{T}}$ one has $\uparrow A = \uparrow \{A\}$ and $\uparrow A \in \mathcal{F}_s^{\mathcal{T}}$.*

(iii) *For $A, B \in \mathbb{T}^{\mathcal{T}}$ one has $\uparrow A \sqcup \uparrow B = \uparrow(A \cap B)$.*

(iv) *For $X \in \mathcal{F}_s^{\mathcal{T}}$ one has*

$$\begin{aligned} X &= \bigsqcup \{\uparrow A \mid A \in X\} &&= \bigsqcup \{\uparrow A \mid \uparrow A \subseteq X\} \\ &= \bigcup \{\uparrow A \mid A \in X\} &&= \bigcup \{\uparrow A \mid \uparrow A \subseteq X\}. \end{aligned}$$

(v) *$\{\uparrow A \mid A \in \mathbb{T}^{\mathcal{T}}\} \cup \{\emptyset\}$ is the set of finite elements of $\mathcal{F}_s^{\mathcal{T}}$.*

Proof. Immediate. ■

15.4.7. Remark. The items 15.1.9-15.2.10 and 15.4.1-15.4.6 are about type theories, but can be translated immediately to structures and if no $\to$ are involved to meet-semi lattices. For example Proposition 15.1.13 also holds for a proper type structure, hence it holds for Scott, Park, CDZ, HR, DHM, BCD, AO, HL and CDV considered as type structures. Also 15.1.14-15.1.20 immediately yield corresponding valid statements for the corresponding type structures, though the proof for the type theories cannot be translated to proofs for the type structures because they are by induction on the syntactic generation of $\mathbb{T}$ or $\leq$. Also 15.2.4-15.2.10 hold for type structures, as follows immediately from Propositions 15.3.5 and 15.3.6. Finally 15.4.1-15.4.6 can be translated immediately to type structures and meet semi-lattices. Therefore in the following chapters everywhere the type theories may be translated to type structures (or if no $\to$ is involved to meet semi-lattices). In Chapter 17 we work directly with meet semi-lattices and type structures and not with type theories, because there a partial order is needed.

15.5. Exercises 31.10.2006:581

15.5.1. Show that $\Gamma, x{:}\top \vdash^{\mathcal{T}}_{\cap\top} M : A \;\Rightarrow\; \Gamma \vdash^{\mathcal{T}}_{\cap\top} M : A$.

15.5.2. The system $\mathtt{K}$ and the type assignment system $\lambda^{\mathtt{K}}_{\cap}$ of Krivine [1990] are CD and $\lambda^{\mathrm{CD}}_{\cap}$, but with rule $(\leq)$ replaced by

$$(\cap\mathrm{E}) \quad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B}$$

Similarly $\mathtt{K}^{\top}$ and $\lambda^{\mathtt{K}^{\top}}_{\cap\top}$ are CDS and $\lambda^{\mathrm{CDS}}_{\cap\top}$, with $(\leq)$ replaced by $(\cap\mathrm{E})$. Show that

(i) $\Gamma \vdash^{\mathtt{K}} M : A \quad\Leftrightarrow\quad \Gamma \vdash^{\mathrm{CD}}_{\cap} M : A$.

(ii) $\Gamma \vdash^{\mathtt{K}^{\top}} M : A \quad\Leftrightarrow\quad \Gamma \vdash^{\mathrm{CDS}}_{\cap\top} M : A$.

15.5.3. (i) Show that $\lambda x.xxx$ and $(\lambda x.xx)\mathsf{I}$ are typable in system $\mathtt{K}$.
(ii) Show that all closed terms in normal forms are typable in system $\mathtt{K}$.

15.5.4. Show the following:

(i) $\vdash^{\mathtt{K}} \lambda z.\mathsf{KI}(zz) : (A{\to}B) \cap A{\to}C{\to}C$.

(ii) $\vdash^{\mathtt{K}^{\top}} \lambda z.\mathsf{KI}(zz) : \top{\to}C{\to}C$.

(iii) $\vdash^{\mathrm{BCD}}_{\cap\top} \lambda z.\mathsf{KI}(zz) : \top{\to}(A{\to}B \cap C){\to}A{\to}B$.

15.5.5. For $\mathcal{T}$ a type theory, $M, N \in \Lambda$ and $x \notin \mathrm{dom}(\Gamma)$ show

(i) $\Gamma \vdash^{\mathcal{T}}_{\cap} M : A \quad\Rightarrow\quad \Gamma \vdash^{\mathcal{T}}_{\cap} M[x{:}=N] : A$;
(ii) $\Gamma \vdash^{\mathcal{T}}_{\cap\top} M : A \quad\Rightarrow\quad \Gamma \vdash^{\mathcal{T}}_{\cap\top} M[x{:}=N] : A$.

15.5.6. Show that

$$M \text{ is a closed term } \Rightarrow \vdash^{\mathrm{Park}}_{\cap\top} M : \omega.$$

Later we will show the converse (Theorem 18.3.22).

15.5.7. Prove that for all types $A \in \mathbb{T}^{\mathrm{AO}}$ there is an n such that

$$\top^n \to \top \leq_{\mathrm{AO}} A.$$

15.5.8. Prove that if $(\omega\varphi), (\varphi \to \omega)$ and $(\omega \to \varphi)$ are axioms in $\mathcal{T}$, then for all M in normal form $\{x_1 : \omega, \ldots, x_n : \omega\} \vdash^{\mathcal{T}} M : \varphi$, where $\{x_1, \ldots, x_n\} \supseteq \mathrm{FV}(M)$.

15.5.9. Let $\mathcal{D} = \langle D, \cdot \rangle$ be an applicative structure, i.e. a set with an arbitrary binary operation on it. For $X, Y \subset D$ define

$$X \to Y = \{d \in D \mid \forall e \in X. d \cdot e \in Y\}.$$

Consider $(\mathcal{P}(D), \to, \subseteq, \cap, D)$, where $\mathcal{P}(D)$ is the power set of D, $\subseteq$ and $\cap$ are the usual set theoretic notions and D is the top of $\mathcal{P}(D)$. Show

- $(\mathcal{P}(\mathcal{D}), \to, \subseteq, \cap)$ is a proper type structure.
- $\mathcal{D} = \mathcal{D} \to \mathcal{D}$.
- $(\mathcal{P}(\mathcal{D}), \to, \subseteq, \cap, \mathcal{D})$ is a natural type structure.