Chapter 2

Properties

2.1. First properties

In this section we will treat simple properties of the various systems $\lambda_\to$. Deeper properties—like strong normalization of typeable terms—will be considered in Section 2.2.

Properties of $\lambda_\to^{\mathrm{Cu}}$, $\lambda_\to^{\mathrm{Ch}}$ and $\lambda_\to^{\mathrm{dB}}$

Unless stated otherwise, properties stated for $\lambda_\to$ apply to both systems.

2.1.1. PROPOSITION (Weakening lemma for $\lambda_\to$).
Suppose $\Gamma \vdash M : A$ *and* Γ' *is a basis with* $\Gamma \subseteq \Gamma'$. *Then* $\Gamma' \vdash M : A$.

PROOF. By induction on the derivation of $\Gamma \vdash M : A$. ■

2.1.2. LEMMA (Free variable lemma). (i) *Suppose* $\Gamma \vdash M : A$. *Then* $FV(M) \subseteq \mathrm{dom}(\Gamma)$.
(ii) *If* $\Gamma \vdash M : A$, *then* $\Gamma \upharpoonright \mathrm{FV}(M) \vdash A : M$, *where for a set* X *of variables one has* $\Gamma \upharpoonright \mathrm{FV}(M) = \{x{:}A \in \Gamma \mid x \in X\}$.

PROOF. (i), (ii) By induction on the generation of $\Gamma \vdash M : A$. ■

The following result is related to the fact that the system $\lambda_\to$ is 'syntax directed', i.e. statements $\Gamma \vdash M : A$ have a unique proof.

2.1.3. PROPOSITION (Generation lemma for $\lambda_\to^{\mathrm{Cu}}$).

$$\begin{array}{lrcl} \text{(i)} & \Gamma \vdash x : A & \Rightarrow & (x{:}A) \in \Gamma. \\ \text{(ii)} & \Gamma \vdash MN : A & \Rightarrow & \exists B \in \mathbb{T}\ [\Gamma \vdash M : B \to A\ \&\ \Gamma \vdash N : B]. \\ \text{(iii)} & \Gamma \vdash \lambda x.M : A & \Rightarrow & \exists B, C \in \mathbb{T}\ [A \equiv B \to C\ \&\ \Gamma, x{:}B \vdash M : C]. \end{array}$$

PROOF. (i) Suppose $\Gamma \vdash x : A$ holds in $\lambda_\to$. The last rule in a derivation of this statement cannot be an application or an abstraction, since x is not of the right form. Therefore it must be an axiom, i.e. $(x{:}A) \in \Gamma$.
(ii), (iii) The other two implications are proved similarly. ■

2.1.4. Proposition (Generation lemma for $\lambda_{\to}^{\mathrm{dB}}$).

$$\begin{array}{lrcl} \text{(i)} & \Gamma \vdash x : A & \Rightarrow & (x{:}A) \in \Gamma. \\ \text{(ii)} & \Gamma \vdash MN : A & \Rightarrow & \exists B \in \mathbb{T}\, [\Gamma \vdash M : B{\to}A \ \&\ \Gamma \vdash N : B]. \\ \text{(iii)} & \Gamma \vdash \lambda x{:}B.M : A & \Rightarrow & \exists C \in \mathbb{T}\, [A \equiv B{\to}C \ \&\ \Gamma, x{:}B \vdash M : C]. \end{array}$$

Proof. Similarly. ■

2.1.5. Proposition (Generation lemma for $\lambda_{\to}^{\mathrm{Ch}}$).

$$\begin{array}{lrcl} \text{(i)} & x^B \in \Lambda_{\to}^{\mathrm{Ch}}(A) & \Rightarrow & B = A. \\ \text{(ii)} & (MN) \in \Lambda_{\to}^{\mathrm{Ch}}(A) & \Rightarrow & \exists B \in \mathbb{T}.[M \in \Lambda_{\to}^{\mathrm{Ch}}(B{\to}A) \ \&\ N \in \Lambda_{\to}^{\mathrm{Ch}}(B)]. \\ \text{(iii)} & (\lambda x^B.M) \in \Lambda_{\to}^{\mathrm{Ch}}(A) & \Rightarrow & \exists C \in \mathbb{T}.[A = (B{\to}C) \ \&\ M \in \Lambda_{\to}^{\mathrm{Ch}}(C)]. \end{array}$$

Proof. As before. ■

The following two results hold for $\lambda_{\to}^{\mathrm{Cu}}$ and $\lambda_{\to}^{\mathrm{dB}}$. Variants already have been proved for $\lambda_{\to}^{\mathrm{Ch}}$, Propositions 1.4.2 and 1.4.4(iii).

2.1.6. Proposition (Substitution lemma for $\lambda_{\to}^{\mathrm{Cu}}$ and $\lambda_{\to}^{\mathrm{dB}}$).
(i) $\Gamma, x{:}A \vdash M : B \ \&\ \Gamma \vdash N : A \ \Rightarrow\ \Gamma \vdash M[x{:}= N] : B$.
(ii) $\Gamma \vdash M : A \ \Rightarrow\ \Gamma[\alpha := B] \vdash M : A[\alpha := B]$.

Proof. The proof will be given for $\lambda_{\to}^{\mathrm{Cu}}$, for $\lambda_{\to}^{\mathrm{dB}}$ it is similar.
(i) By induction on the derivation of $\Gamma, x{:}A \vdash M : B$. Write $P^* \equiv P[x{:}= N]$.
Case 1. $\Gamma, x{:}A \vdash M : B$ is an axiom, hence $M \equiv y$ and $(y{:}B) \in \Gamma \cup \{x{:}A\}$.
Subcase 1.1. $(y{:}B) \in \Gamma$. Then $y \not\equiv x$ and $\Gamma \vdash M^* \equiv y[x{:}N] \equiv y : B$.
Subcase 1.2. $y{:}B \equiv x{:}A$. Then $y \equiv x$ and $B \equiv A$, hence $\Gamma \vdash M^* \equiv N : A \equiv B$.
Case 2. $\Gamma, x{:}A \vdash M : B$ follows from $\Gamma, x{:}A \vdash F : C{\to}B$, $\Gamma, x{:}A \vdash G : C$ and $FG \equiv M$. By the induction hypothesis one has $\Gamma \vdash F^* : C{\to}B$ and $\Gamma \vdash G^* : C$. Hence $\Gamma \vdash (FG)^* \equiv F^*G^* : B$.
Case 3. $\Gamma, x{:}A \vdash M : B$ follows from $\Gamma, x{:}A, y{:}D \vdash G : E$, $B \equiv D{\to}E$ and $\lambda y.G \equiv M$. By the induction hypothesis $\Gamma, y{:}D \vdash G^* : E$, hence $\Gamma \vdash (\lambda y.G)^* \equiv \lambda y.G^* : D{\to}E \equiv B$.
(ii) Similarly. ■

2.1.7. Proposition (Subject reduction property for $\lambda_{\to}^{\mathrm{Cu}}$ and $\lambda_{\to}^{\mathrm{dB}}$). *Suppose $M \twoheadrightarrow_{\beta\eta} M'$. Then $\Gamma \vdash M : A \ \Rightarrow\ \Gamma \vdash M' : A$.*

Proof. The proof will be given for $\lambda_{\to}^{\mathrm{dB}}$, for $\lambda_{\to}^{\mathrm{Cu}}$ it is similar. Suppose $\Gamma \vdash M : A$ and $M \to M'$ in order to show that $\Gamma \vdash M' : A$; then the result follows by induction on the derivation of $\Gamma \vdash M : A$.
Case 1. $\Gamma \vdash M : A$ is an axiom. Then M is a variable, contradicting $M \to M'$. Hence this case cannot occur.
Case 2. $\Gamma \vdash M : A$ is $\Gamma \vdash FN : A$ and is a direct consequence of $\Gamma \vdash F : B{\to}A$ and $\Gamma \vdash N : B$. Since $FN \equiv M \to M'$ we can have three subcases.

Subcase 2.1. $M' \equiv F'N$ with $F \to F'$.

Subcase 2.2. $M' \equiv FN'$ with $N \to N'$.

In these two subcases it follows by the induction hypothesis that $\Gamma \vdash M' : A$.

Subcase 2.3. $F \equiv \lambda x{:}B.G$ and $M' \equiv G[x{:}= N]$. Since

$$\Gamma \vdash \lambda x.G : B{\to}A \ \&\ \Gamma \vdash N : B$$

it follows by the generation lemma 2.1.3 for $\lambda_\to$ that

$$\Gamma, x{:}B \vdash G : A \ \&\ \Gamma \vdash N : B.$$

Therefore by the substitution lemma 2.1.6 for $\lambda_\to$ it follows that
$\Gamma \vdash G[x{:}= N] : A$, i.e. $\Gamma \vdash M' : A$.

Case 3. $\Gamma \vdash M : A$ is $\Gamma \vdash \lambda x{:}B.N : B{\to}C$ and follows from $\Gamma, x{:}B \vdash N : C$. Since $M \to M'$ we have $M' \equiv \lambda x{:}B.N'$ with $N \to N'$. By the induction hypothesis one has $\Gamma, x{:}B \vdash N' : C$, hence $\Gamma \vdash \lambda x{:}B.N' : B{\to}C$, i.e. $\Gamma \vdash M' : A$. ∎

The following result also holds for $\lambda_\to^{\mathrm{Ch}}$ and $\lambda_\to^{\mathrm{dB}}$, Exercise 2.5.4.

2.1.8. Corollary (Church-Rosser Theorem for $\lambda_\to^{\mathrm{Cu}}$). *On typable terms of $\lambda_\to^{\mathrm{Cu}}$ the Church-Rosser theorem holds for the notions of reduction $\twoheadrightarrow_{\boldsymbol{\beta}}$ and $\twoheadrightarrow_{\boldsymbol{\beta\eta}}$.*

(i) *Let $M, N \in \Lambda_\to^\Gamma(A)$. Then*

$$M =_{\boldsymbol{\beta}(\boldsymbol{\eta})} N \ \Rightarrow\ \exists Z \in \Lambda_\to^\Gamma(A).M \twoheadrightarrow_{\boldsymbol{\beta}(\boldsymbol{\eta})} Z \ \&\ N \twoheadrightarrow_{\boldsymbol{\beta}(\boldsymbol{\eta})} Z.$$

(ii) *Let $M, N_1, N_2 \in \Lambda_\to^\Gamma(A)$. Then*

$$M \twoheadrightarrow_{\boldsymbol{\beta\eta}} N_1 \ \&\ M \twoheadrightarrow_{\boldsymbol{\beta}(\boldsymbol{\eta})} N_2 \ \Rightarrow\ \exists Z \in \Lambda_\to^\Gamma(A).N_1 \twoheadrightarrow_{\boldsymbol{\beta}(\boldsymbol{\eta})} Z \ \&\ N_2 \twoheadrightarrow_{\boldsymbol{\beta}(\boldsymbol{\eta})} Z.$$

Proof. By the Church-Rosser theorems for $\twoheadrightarrow_{\boldsymbol{\beta}}$ and $\twoheadrightarrow_{\boldsymbol{\beta\eta}}$ on untyped terms, Theorem 1.1.7, and Proposition 2.1.7. ∎

The following property of uniqueness of types only holds for the Church and de Bruijn versions of $\lambda_\to$. It is instructive to find out where the proof brakes down for $\lambda_\to^{\mathrm{Cu}}$ and also that the two contexts in (ii) should be the same.

2.1.9. Proposition (Unicity of types for $\lambda_\to^{\mathrm{Ch}}$ and $\lambda_\to^{\mathrm{dB}}$).

(i) $M \in \Lambda_\to^{\mathrm{Ch}}(A) \ \&\ M \in \Lambda_\to^{\mathrm{Ch}}(B) \quad \Rightarrow \quad A = B.$

(ii) $\Gamma \vdash_{\lambda_\to}^{\mathrm{dB}} M : A \ \&\ \Gamma \vdash_{\lambda_\to}^{\mathrm{dB}} M : B \quad \Rightarrow \quad A = B.$

Proof. (i), (ii) By induction on the structure of M, using the generation lemma 2.1.4. ∎

Normalization

For several applications, for example for the problem to find all possible inhabitants of a given type, we will need the weak normalization theorem, stating that all typable terms do have a $\boldsymbol{\beta\eta}$-nf (normal form). The result is valid for all versions of $\lambda_{\rightarrow}$ and *a fortiori* for the subsystems $\lambda^{o}_{\rightarrow}$. The proof is due to Turing and is published posthumously in Gandy [1980]. In fact all typable terms in these systems are $\boldsymbol{\beta\eta}$ strongly normalizing, which means that all $\boldsymbol{\beta\eta}$-reductions are terminating. This fact requires more work and will be proved in §12.2.

The notion of 'abstract reduction system', see Klop [1992], is useful for the understanding of the proof of the normalization theorem.

2.1.10. Definition. (i) An *abstract reduction system* is a pair $(X, \rightarrow_R)$, where X is a set and $\rightarrow_R$ is a binary relation on X.

(ii) An element $x \in X$ is said to be in R-normal form (R-nf) if for no $y \in X$ one has $x \rightarrow_R y$.

(iii) (X, R) is called *weakly normalizing* (R-WN, or simply WN) if every element has an R-nf.

(iv) (X, R) is said to be *strongly normalizing* (R-SN, or simply SN) if every R-reduction path

$$x_0 \rightarrow_R x_1 \rightarrow_R x_2 \rightarrow_R \dots.$$

is finite.

2.1.11. Definition. (i) A *multiset over nat* can be thought of as a generalized set S in which each element may occur more than once. For example

$$S = \{3, 3, 1, 0\}$$

is a multiset. We say that 3 occurs in S with multiplicity 2; that 1 has multiplicity 1; etcetera.

More formally, the above multiset S can be identified with a function $f \in \mathbb{N}^{\mathbb{N}}$ that is almost everywhere 0, except

$$f(0) = 1, f(1) = 1, f(3) = 2.$$

This S is finite if f has *finite support*, where

$$\texttt{support}(f) = \{x \in \mathbb{N} \mid f(x) \neq 0\}.$$

(ii) Let $\mathcal{S}(\mathbb{N})$ be the collection of all finite multisets over $\mathbb{N}$. $\mathcal{S}(\mathbb{N})$ can be identified with $\{f \in \mathbb{N}^{\mathbb{N}} \mid \texttt{support}(f) \text{ is finite}\}$.

2.1.12. Definition. Let $S_1, S_2 \in \mathcal{S}(\mathbb{N})$. Write

$$S_1 \rightarrow_{\mathcal{S}} S_2$$

if S_2 results from S_1 by replacing some elements (just one occurrence) by finitely many lower elements (in the usual ordering of $\mathbb{N}$). For example

$$\{3, \underline{3}, 1, 0\} \to_{\mathcal{S}} \{3, \underline{2, 2, 2, 1}, 1, 0\}.$$

2.1.13. Lemma. *We define a particular (non-deterministic) reduction strategy F on $\mathcal{S}(\mathbb{N})$. A multi-set S is contracted to $F(S)$ by taking a maximal element $n \in S$ and replacing it by finitely many numbers $< n$. Then F is a normalizing reduction strategy, i.e. for every $S \in \mathcal{S}(\mathbb{N})$ the $\mathcal{S}$-reduction sequence*

$$S \to_{\mathcal{S}} F(S) \to_{\mathcal{S}} F^2(S) \to_{\mathcal{S}} \dots$$

is terminating.

Proof. By induction on the highest number n occuring in S. If $n = 0$, then we are done. If $n = k+1$, then we can successively replace in S all occurrences of n by numbers $\leq k$ obtaining S_1 with maximal number $\leq k$. Then we are done by the induction hypothesis. ■

In fact $(\mathcal{S}(\mathbb{N}), \twoheadrightarrow_{\mathcal{S}})$ is SN. Although we do not strictly need this fact, we will give even two proofs of it. In the first place it is something one ought to know; in the second place it is instructive to see that the result does not imply that $\lambda_{\to}$ satisfies SN.

2.1.14. Lemma. *The reduction system $(\mathcal{S}(\mathbb{N}), \to_{\mathcal{S}})$ is SN.*

We will give two proofs of this lemma. The first one uses ordinals; the second one is from first principles.

Proof$_1$. Assign to every $S \in \mathcal{S}(\mathbb{N})$ an ordinal $\#S < \omega^\omega$ as suggested by the following examples.

$$\begin{aligned} \#\{3, 3, 1, 0, 0, 0\} &= 2\omega^3 + \omega + 3; \\ \#\{3, 2, 2, 2, 1, 1, 0\} &= \omega^3 + 3\omega^2 + 2\omega + 1. \end{aligned}$$

More formally, if S is represented by $f \in \mathbb{N}^{\mathbb{N}}$ with finite support, then

$$\#S = \Sigma_{i \in \mathbb{N}} f(i) \cdot \omega^i.$$

Notice that

$$S_1 \to_{\mathcal{S}} S_2 \;\Rightarrow\; \#S_1 > \#S_2$$

(in the example because $\omega^3 > 3\omega^2 + \omega$). Hence by the well-foundedness of the ordinals the result follows. ■$_1$

Proof$_2$. Define

$$\begin{aligned} \mathcal{F}_k &= \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n \geq k \, f(n) = 0\}; \\ \mathcal{F} &= \cup_{k \in \mathbb{N}} \mathcal{F}_k. \end{aligned}$$

The set $\mathcal{F}$ is the set of functions with finite support. Define on $\mathcal{F}$ the relation $>$ corresponding to the relation $\rightarrow_{\mathcal{S}}$ for the formal definition of $\mathcal{S}(\mathbb{N})$.

$$f > g \iff f(k) > g(k), \text{ where } k \in \mathbb{N} \text{ is largest such that } f(k) \neq g(k).$$

It is easy to see that $(\mathcal{F}, >)$ is a linear ordering. We will show that it is even a well-ordering, i.e. for every non-empty set $X \subseteq \mathcal{F}$ there is a least element $f_0 \in X$. This implies that there are no infinite descending chains in $\mathcal{F}$.

To show this claim it suffices to prove that each $\mathcal{F}_k$ is well-ordered, since

$$\ldots > (\mathcal{F}_{k+1} \setminus \mathcal{F}_k) > \mathcal{F}_k$$

element-wise. This will be proved by induction on k. If $k = 0$, then this is trivial, since $\mathcal{F}_0 = \{\lambda\!\!\lambda n.0\}$. Now assume (induction hypothesis) that $\mathcal{F}_k$ is well-ordered in order to show the same for $\mathcal{F}_{k+1}$. Let $X \subseteq \mathcal{F}_{k+1}$ be non-empty. Define

$$\begin{aligned} X(k) &= \{f(k) \mid f \in X\} \subseteq \mathbb{N}; \\ X_k &= \{f \in X \mid f(k) \text{ minimal in } X(k)\} \subseteq \mathcal{F}_{k+1}; \\ X_k|k &= \{g \in \mathcal{F}_k \mid \exists f \in X_k\, f|k = g\} \subseteq \mathcal{F}_k, \end{aligned}$$

where

$$\begin{aligned} f|k(i) &= f(i), && \text{if } i < k; \\ &= 0, && \text{else.} \end{aligned}$$

By the induction hypothesis $X_k|k$ has a least element g_0. Then $g_0 = f_0|k$ for some $f_0 \in X_k$. This f_0 is then the least element of X_k and hence of X. $\blacksquare_2$

2.1.15. Remark. The second proof shows in fact that if $(D, >)$ is a well-ordered set, then so is $(\mathcal{S}(D), >)$, defined analogously to $(\mathcal{S}(\mathbb{N}), >)$. In fact the argument can be carried out in Peano Arithmetic, showing

$$\vdash_{\mathbf{PA}} \mathtt{TI}(\alpha) \rightarrow \mathtt{TI}(\alpha^{\omega}),$$

where $\mathtt{TI}(\alpha)$ is the principle of transfinite induction for the ordinal α. Since $\mathtt{TI}(\omega)$ is in fact ordinary induction we have in PA

$$\mathtt{TI}(\omega),\ \mathtt{TI}(\omega^{\omega}),\ \mathtt{TI}(\omega^{(\omega^{\omega})}), \ldots.$$

This implies that the proof of $\mathtt{TI}(\alpha)$ can be carried out in Peano Arithmetic for every $\alpha < \epsilon_0$. Gentzen [1936] shows that $\mathtt{TI}(\epsilon_0)$, where $\epsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$, cannot be carried out in PA.

In order to prove the $\lambda_{\rightarrow}$ is WN it suffices to work with $\lambda_{\rightarrow}^{\mathrm{Ch}}$. We will use the following notation. We write terms with extra type information, decorating each subterm with its type. For example, instead of $(\lambda x^A.M)N \in \mathtt{term}_B$ we write $(\lambda x^A.M^B)^{A\rightarrow B}N^A$.

2.1.16. Definition. (i) Let $R \equiv (\lambda x^A.M^B)^{A\to B}N^A$ be a redex. The *depth* of R, notation $\#R$, is defined as follows.

$$\#R = \#(A\to B)$$

where # on types is defined inductively by

$$\begin{aligned} \#\alpha &= 0; \\ \#(A\to B) &= \max(\#A, \#B) + 1. \end{aligned}$$

(ii) To each M in $\lambda_{\to}^{\mathrm{Ch}}$ we assign a multi-set S_M as follows

$$S_M = \{\#R \mid R \text{ is a redex occurrence in } M\},$$

with the understanding that the multiplicity of R in M is copied in S_M.

In the following example we study how the contraction of one redex can duplicate other redexes or create new redexes.

2.1.17. Example. (i) Let R be a redex occurrence in a typed term M. Assume

$$M \xrightarrow{R}_{\boldsymbol\beta} N,$$

i.e. N results form M by contracting R. This contraction can duplicate other redexes. For example (we write $M[P]$, or $M[P,Q]$ to display subterms of M)

$$(\lambda x.M[x,x])R_1 \to_{\boldsymbol\beta} M[R_1,R_1]$$

duplicates the other redex R_1.

(ii) (J.J. Lévy [1978]) Contraction of a $\boldsymbol\beta$-redex may also create new redexes. For example

$$\begin{aligned} (\lambda x^{A\to B}.M[x^{A\to B}P^A]^C)^{(A\to B)\to C}(\lambda y^A.Q^B) &\to_{\boldsymbol\beta} M[(\lambda y^A.Q^B)^{A\to B}P^A]^C; \\ (\lambda x^A.(\lambda y^B.M[x^A,y^B]^C)^{B\to C})^{A\to(B\to C)}P^AQ^B &\to_{\boldsymbol\beta} (\lambda y^B.M[P^A,y^B]^C)^{B\to C}Q^B; \\ (\lambda x^{A\to B}.x^{A\to B})^{(A\to B)\to(A\to B)}(\lambda y^A.P^B)^{A\to B}Q^A &\to_{\boldsymbol\beta} (\lambda y^A.P^B)^{A\to B}Q^A. \end{aligned}$$

2.1.18. Lemma. *Assume $M \xrightarrow{R}_{\boldsymbol\beta} N$ and let R_1 be a created redex in N. Then $\#R > \#R_1$.*

Proof. In Lévy [1978] it is proved that the three ways of creating redexes in example 2.1.17(ii) are the only possibilities. For a proof do exercise 14.5.3 in B[1984]. In each of three cases we can inspect that the statement holds. ∎

2.1.19. Theorem (Weak normalization theorem for $\lambda_\to$). *If $M \in \Lambda$ is typable in $\lambda_\to$, then M is $\boldsymbol{\beta\eta}$-WN, i.e. has a $\boldsymbol{\beta\eta}$-nf.*

Proof. By Proposition 1.4.9(ii) it suffices to show this for terms in $\lambda^{\mathrm{Ch}}_{\rightarrow}$. Note $\boldsymbol{\eta}$-reductions decreases the length of a term; moreover, for $\boldsymbol{\beta}$-normal terms $\boldsymbol{\eta}$-contractions do not create $\boldsymbol{\beta}$-redexes. Therefore in order to establish $\boldsymbol{\beta\eta}$-WN it is sufficient to prove that M has a $\boldsymbol{\beta}$-nf.

Define the following $\boldsymbol{\beta}$-reduction strategy F. If M is in nf, then $F(M) = M$. Otherwise, let R be the *rightmost redex of maximal depth* n in M. Then

$$F(M) = N$$

where $M \xrightarrow{R}_{\boldsymbol{\beta}} N$. Contracting a redex can only duplicate other redexes that are to the right of that redex. Therefore by the choice of R there can only be redexes of M duplicated in $F(M)$ of depth $< n$. By lemma 2.1.18 redexes created in $F(M)$ by the contraction $M \to_{\boldsymbol{\beta}} F(M)$ are also of depth $< n$. Therefore in case M is not in $\boldsymbol{\beta}$-nf we have

$$S_M \to_{\mathcal{S}} S_{F(M)}.$$

Since $\to_{\mathcal{S}}$ is SN, it follows that the reduction

$$M \to_{\boldsymbol{\beta}} F(M) \to_{\boldsymbol{\beta}} F^2(M) \to_{\boldsymbol{\beta}} F^3(M) \to_{\boldsymbol{\beta}} \ldots$$

must terminate in a $\boldsymbol{\beta}$-nf. ■

For $\boldsymbol{\beta}$-reduction this weak normalization theorem was first proved by Turing, see Gandy [1980b]. The proof does not really need SN for $\mathcal{S}$-reduction. One may also use the simpler result lemma 2.1.13.

It is easy to see that a different reduction strategy does not yield a $\mathcal{S}$-reduction chain. For example the two terms

$$\begin{array}{c} (\lambda x^A.y^{A\to A\to A}x^Ax^A)^{A\to A}((\lambda x^A.x^A)^{A\to A}x^A) \quad \to_{\boldsymbol{\beta}} \\ y^{A\to A\to A}((\lambda x^A.x^A)^{A\to A}x^A)((\lambda x^A.x^A)^{A\to A}x^A) \end{array}$$

give the multisets $\{1,1\}$ and $\{1,1\}$. Nevertheless, SN does hold for all systems $\lambda_\rightarrow$, as will be proved in Section 2.2. It is an open problem whether ordinals can be assigned in a natural and simple way to terms of $\lambda_\rightarrow$ such that

$$M \to_{\boldsymbol{\beta}} N \ \Rightarrow\ \mathtt{ord}(M) > \mathtt{ord}(N).$$

See Howard [1970] and de Vrijer [1987].

Applications of normalization

We will prove that normal terms inhabiting the represented data types (Bool, Nat, Σ^* and T_B) are standard, i.e. correspond to the intended elements. From WN for $\lambda_\rightarrow$ and the subject reduction theorem it then follows that all inhabitants of the mentioned data types are standard.

2.1.20. Proposition. *Let* $M \in \Lambda$ *be in nf. Then* $M \equiv \lambda x_1 \cdots x_n.yM_1 \ldots M_m$, *with* $n, m \geq 0$ *and the* $M_1, \ldots, M_m$ *again in nf.*

PROOF. By induction on the structure of M. See Barendregt [1984], proposition 8.3.8 for some details if necessary. ■

2.1.21. PROPOSITION. *Let* $\mathsf{Bool} \equiv \mathsf{Bool}_\alpha$, *with* α *a type variable. Then for* M *in nf one has*

$$\vdash M : \mathsf{Bool} \;\Rightarrow\; M \in \{\mathsf{true}, \mathsf{false}\}.$$

PROOF. By repeated use of proposition 2.1.20, the free variable lemma 2.1.2 and the generation lemma for $\lambda_{\rightarrow}^{\mathrm{Cu}}$, proposition 2.1.3, one has the following chain of arguments.

$$\begin{aligned} \vdash M : \alpha{\to}\alpha{\to}\alpha \quad &\Rightarrow \quad M \equiv \lambda x.M_1 \\ &\Rightarrow \quad x{:}\alpha \vdash M_1 : \alpha{\to}\alpha \\ &\Rightarrow \quad M_1 \equiv \lambda y.M_2 \\ &\Rightarrow \quad x{:}\alpha, y{:}\alpha \vdash M_2 : \alpha \\ &\Rightarrow \quad M_2 \equiv x \text{ or } M_2 \equiv y. \end{aligned}$$

So $M \equiv \lambda xy.x \equiv \mathsf{true}$ or $M \equiv \lambda xy.y \equiv \mathsf{false}$. ■

2.1.22. PROPOSITION. *Let* $\mathsf{Nat} \equiv \mathsf{Nat}_\alpha$. *Then for* M *in nf one has*

$$\vdash M : \mathsf{Nat} \;\Rightarrow\; M \in \{\ulcorner n \urcorner \mid n \in \mathbb{N}\}.$$

PROOF. Again we have

$$\begin{aligned} \vdash M : \alpha{\to}(\alpha{\to}\alpha){\to}\alpha \quad &\Rightarrow \quad M \equiv \lambda x.M_1 \\ &\Rightarrow \quad x{:}\alpha \vdash M_1 : (\alpha{\to}\alpha){\to}\alpha \\ &\Rightarrow \quad M_1 \equiv \lambda f.M_2 \\ &\Rightarrow \quad x{:}\alpha, f{:}\alpha{\to}\alpha \vdash M_2 : \alpha. \end{aligned}$$

Now we have

$$\begin{aligned} x{:}\alpha, f{:}\alpha{\to}\alpha \vdash M_2 : \alpha \quad \Rightarrow \quad & [M_2 \equiv x \vee \\ & [M_2 \equiv f M_3 \;\&\; x{:}\alpha, f{:}\alpha{\to}\alpha \vdash M_3 : \alpha]]. \end{aligned}$$

Therefore by induction on the structure of M_2 it follows that

$$x{:}\alpha, f{:}\alpha{\to}\alpha \vdash M_2 : \alpha \;\Rightarrow\; M_2 \equiv f^n(x),$$

with $n \geq 0$. So $M \equiv \lambda xf.f^n(x) \equiv \ulcorner n \urcorner$. ■

2.1.23. PROPOSITION. *Let* $\mathsf{Sigma}^* \equiv \mathsf{Sigma}^*_\alpha$. *Then for* M *in nf one has*

$$\vdash M : \mathsf{Sigma}^* \;\Rightarrow\; M \in \{\underline{w} \mid w \in \Sigma^*\}.$$

Proof. Again we have

$$\begin{array}{rcl} \vdash M : \alpha{\to}(\alpha{\to}\alpha)^k{\to}\alpha & \Rightarrow & M \equiv \lambda x.N \\ & \Rightarrow & x{:}\alpha \vdash N : (\alpha{\to}\alpha)^k{\to}\alpha \\ & \Rightarrow & N \equiv \lambda a_1.N_1 \;\&\; x{:}\alpha, a_1{:}\alpha{\to}\alpha \vdash N_1 : (\alpha{\to}\alpha)^{k-1}{\to}\alpha \\ & \cdots & \\ & \Rightarrow & N \equiv \lambda a_1 \cdots a_k.N \;\&\; x{:}\alpha, a_1, \ldots, a_k{:}\alpha{\to}\alpha \vdash N_k : \alpha \\ & \Rightarrow & [N_k \equiv x \vee \\ & & [N_k \equiv a_{i_j} N'{}_k \;\&\; x{:}\alpha, a_1, \ldots, a_k{:}\alpha{\to}\alpha \vdash N_k{}' : \alpha]] \\ & \Rightarrow & N_k \equiv a_{i_1}(a_{i_2}(\cdots(a_{i_p}x)\cdot\cdot)) \\ & \Rightarrow & M \equiv \lambda x a_1 \cdots a_k.a_{i_1}(a_{i_2}(\cdots(a_{i_p}x)\cdot\cdot)) \\ & & \equiv \underline{a_{i_1} a_{i_2} \cdots a_{i_p}}. \blacksquare \end{array}$$

Before we can prove that inhabitants of $\mathsf{tree}[\boldsymbol{\beta}]$ are standard, we have to intoduce an auxiliary notion.

2.1.24. Definition. Given $t \in T[b_1, \ldots, b_n]$ define $[t]^{p,l} \in \Lambda$ as follows.

$$\begin{array}{rcl} [b_i]^{p,l} & = & l b_i; \\ [P(t_1, t_2)]^{p,l} & = & p[t_1]^{p,l}[t_2]^{p,l}. \end{array}$$

2.1.25. Lemma. *For $t \in T[b_1, \ldots, b_n]$ we have*

$$[t] =_\beta \lambda pl.[t]^{p,l}.$$

Proof. By induction on the structure of t.

$$\begin{array}{rcll} [b_i] & \equiv & \lambda pl.lb_i & \\ & \equiv & \lambda pl.[b_i]^{p,l}; & \\ [P(t_1,t_2)] & \equiv & \lambda pl.p([t_1]pl)([t_2]pl) & \\ & = & \lambda pl.p[t_1]^{p,l}[t_2]^{p.l}, & \text{by the IH,} \\ & \equiv & \lambda pl.[P(t_1,t_2)]^{p,l}. \blacksquare & \end{array}$$

2.1.26. Proposition. *Let $\mathsf{tree}[\boldsymbol{\beta}] \equiv \mathsf{tree}_\alpha[\boldsymbol{\beta}]$. Then for M in nf one has*

$$b_1, \ldots, b_n{:}\boldsymbol{\beta} \vdash M : \mathsf{tree}[\boldsymbol{\beta}] \;\Rightarrow\; M \in \{[t] \mid t \in T[b_1, \ldots, b_n]\}.$$

PROOF. We have $\vec{b}{:}\boldsymbol{\beta} \vdash M : (\alpha{\to}\alpha{\to}\alpha){\to}(\boldsymbol{\beta}{\to}\alpha){\to}\alpha \quad \Rightarrow$

$$\begin{array}{lll}
\Rightarrow & M \equiv \lambda p.M' & \\
\Rightarrow & \vec{b}{:}\boldsymbol{\beta}, p{:}\alpha{\to}\alpha{\to}\alpha \vdash M' : (\boldsymbol{\beta}{\to}\alpha){\to}\alpha & \\
\Rightarrow & M' \equiv \lambda l.M'' & \\
\Rightarrow & \vec{b}{:}\boldsymbol{\beta}, p{:}(\alpha{\to}\alpha{\to}\alpha), l{:}(\boldsymbol{\beta}{\to}\alpha) \vdash M'' : \alpha & \\
\Rightarrow & M'' \equiv l b_i \ \vee\ [M'' \equiv p M_1 M_2 \ \& & \\
 & \vec{b}{:}\boldsymbol{\beta}, p{:}(\alpha{\to}\alpha{\to}\alpha), l{:}(\boldsymbol{\beta}{\to}\alpha) \vdash M_j : \alpha], & \text{j=1,2,} \\
\Rightarrow & M'' \equiv [t]^{p,l}, \text{ for some } t \in T[\vec{b}], & \\
\Rightarrow & M \equiv \lambda pl.[t]^{p,l} =_\beta [t], & \text{by lemma 2.1.25.} \ \blacksquare
\end{array}$$

2.2. Proofs of strong normalization

We now will give two proofs showing that $\lambda_\to$ is strongly normalizing. The first one is the classical proof due to Tait [1967] that needs little technique, but uses set theoretic comprehension. The second proof due to Statman is elementary, but needs results about reduction.

2.2.1. THEOREM (SN for $\lambda_\to^{\mathrm{Ch}}$). *For all* $A \in \mathbb{T}_\infty$, $M \in \Lambda_\to^{\mathrm{Ch}}(A)$ *one has* $\mathrm{SN}_{\boldsymbol{\beta\eta}}(M)$.

PROOF. We use an induction loading. First we add to $\lambda_\to$ constants $d_\alpha \in \Lambda_\to^{\mathrm{Ch}}(\alpha)$ for each atom α, obtaining $\lambda^+_{\to\mathrm{Ch}}$. Then we prove SN for the extended system. It follows *a fortiori* that the system without the constants is SN.

One first defines for $A \in \mathbb{T}_\infty$ the following class $\mathcal{C}_A$ of *computable* terms of type A. We write SN for $\mathrm{SN}_{\boldsymbol{\beta\eta}}$.

$$\begin{array}{rcl}
\mathcal{C}_\alpha & = & \{M \in \Lambda^\emptyset_{\to\mathrm{Ch}}(\alpha) \mid \mathrm{SN}(M)\}; \\
\mathcal{C}_{A\to B} & = & \{M \in \Lambda^\emptyset_{\to\mathrm{Ch}}(A{\to}B) \mid \forall P \in \mathcal{C}_A . MP \in \mathcal{C}_B\}.
\end{array}$$

Then one defines the classes $\mathcal{C}^*_A$ of terms that are *computable under substitution*

$$\mathcal{C}^*_A = \{M \in \Lambda_{\to\mathrm{Ch}}(A) \mid \forall \vec{Q} \in \mathcal{C}.[M[\vec{x}{:}=\vec{Q}] \in \Lambda^\emptyset_{\to\mathrm{Ch}}(A) \ \Rightarrow\ M[\vec{x}{:}=\vec{Q}] \in \mathcal{C}_A]\}.$$

Write $\mathcal{C}^{(*)} = \bigcup\{\mathcal{C}^{(*)}_A \mid A \in \mathbb{T}(\lambda^+_\to)\}$. For $A = A_1 \to \ldots \to A_n \to \alpha$ define

$$d_A \equiv \lambda x_1{:}A_1 \ldots \lambda x_n{:}A_n.d_\alpha.$$

Then for A one has

$$M \in \mathcal{C}_A \iff \forall \vec{P} \in \mathcal{C}.M\vec{P} \in \mathrm{SN}, \tag{0}$$

$$M \in \mathcal{C}^*_A \iff \forall \vec{P}, \vec{Q} \in \mathcal{C}.M[\vec{x}{:}=\vec{Q}]\vec{P} \in \mathrm{SN}, \tag{1}$$

where the $\vec{P}, \vec{Q}$ should have the right types and $M\vec{P}$ and $M[\vec{x}{:}=\vec{Q}]\vec{P}$ are of type α, respectively. By an easy simultaneous induction on A one can show

$$M \in \mathcal{C}_A \ \Rightarrow\ \mathrm{SN}(M); \tag{2}$$