3.3.37. Example. Let $\mathcal{M}$ be a typed structure. Let $\Delta \subseteq \mathcal{M}$. Write $\Delta(A) = \Delta \cap \mathcal{M}(A)$. Assume that $\Delta(A) \neq \emptyset$ for all $A \in$ and

$$d \in \Delta(A \to B), e \in \Delta(A) \;\Rightarrow\; de \in \Delta(B).$$

Then Δ may fail to be a typed structure because it is not extensional. Equality as binary relation E_o on $\Delta(o) \times \Delta(o)$ induces a binary logical relation E on $\Delta \times \Delta$. Let $\Delta^E = \{d \in \Delta \mid E(d,d)\}$. Then the restriction of E to Δ^E is an applicative congruence and the equivalence classes form a structure. In particular, if $\mathcal{M}$ is a model, then write

$$\Delta^+ = \{d \in \mathcal{M} \mid \exists M \Lambda_o^{\emptyset} \exists d_1 \dots d_n \; [\![M]\!] d_1 \dots d_n = d\}$$

for the applicative closure of Δ. The *Gandy hull* of Δ in $\mathcal{M}$ is the set Δ^{+E}. From the fundamental theorem for logical relations it can be derived that

$$\mathcal{M}_\Delta = \Delta^{+E}/E$$

is a model. This model will be also called the Gandy hull of Δ in $\mathcal{M}$.

3.4. Type reducibility

Remember that a type A is reducible to type B, notation $A \leq_{\beta\eta} B$ if for some closed term $\Phi{:}A \to B$ one has for all closed $M_1, M_2{:}A$

$$M_1 =_{\beta\eta} M_2 \iff \Phi M_1 =_{\beta\eta} \Phi M_2.$$

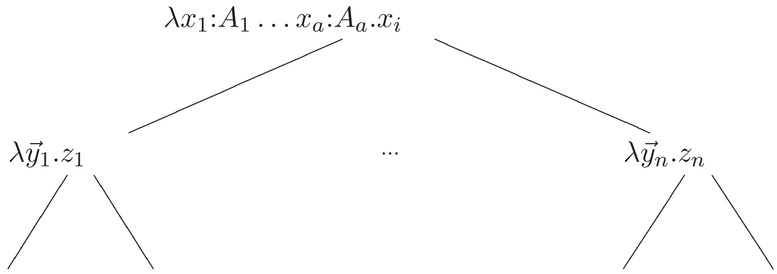
3.4.1. Definition. Write $A \sim_{\boldsymbol{\beta\eta}} B$ iff $A \leq_{\beta\eta} B$ & $B \leq_{\beta\eta} A$.

The reducibility theorem, Statman [1980a], states that there is one type to which all types of $\mathbb{T}(\lambda_\to)$ can be reduced. At first this may seem impossible. Indeed, in a full typed structure $\mathcal{M}$ the cardinality of the sets of higher type increase arbitrarily. So one cannot always have an injection $\mathcal{M}_A \to \mathcal{M}_B$. But reducibility means that one restricts oneself to definable elements (modulo $=_{\beta\eta}$) and then the injections are possible. The proof will occupy 3.4.2-3.4.7. There are four main steps. In order to show that $\Phi M_1 =_{\beta\eta} \Phi M_2 \;\Rightarrow\; M_1 =_{\beta\eta} M_2$ in all cases a (pseudo) inverse Φ^{-1} is used. Pseudo means that sometimes the inverse is not lambda definable, but this is no problem for the implication. Sometimes Φ^{-1} is definable, but the property $\Phi^{-1}(\Phi M) = M$ only holds in an extension of the theory; because the extension will be conservative over $=_{\beta\eta}$ the reducibility follows. Next the type hierarchy theorem, also due to Statman [1980a], will be given. Rather unexpectedly it turns out that under $\leq_{\beta\eta}$ types form a well-ordering of length $\omega + 3$. Finally some consequences of the reducibility theorem will be given, including the 1-section and finite completeness theorems.

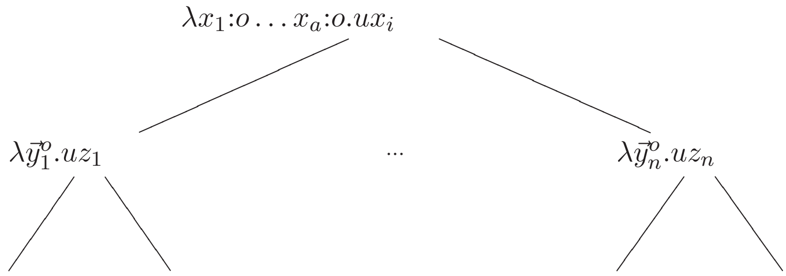
In the first step towards the reducibility theorem it will be shown that every type is reducible to one of rank ≤ 3. The proof is rather syntactic. In order to show that the definable function Φ is 1-1, a non-definable inverse is needed. A warmup exercise for this is 3.6.4.

3.4.2. Proposition. *For every type A there is a type B such that* $\texttt{rank}(B) \leq 3$ *and* $A \leq_{\beta\eta} B$.

Proof. [The intuition behind the construction of the the term Φ responsible for the reducibility is as follows. If M is a term with Böhmtree (see Barendregt [1984])



Now let UM be a term with "Böhmtree" of the form



where all the typed variables are pushed down to type o and the variables u (each occurrence possibly different) takes care that the new term remains typable. From this description it is clear that the u can be chosen in such way that the result has rank ≤ 1. Also that M can be reconstructed from UM so that U is injective. ΦM is just UM with the auxiliary variables bound. This makes it of type with rank ≤ 3. What is less clear is that U and hence Φ are lambda-definable.]

Define inductively for any type A the types $A^\sharp$ and $A^\flat$.

$$\begin{aligned} o^\sharp &= o; \\ o^\flat &= o; \\ (A_1 \to \ldots \to A_a \to o)^\sharp &= o \to A_1^\flat \to \ldots \to A_a^\flat \to o; \\ (A_1 \to \ldots \to A_a \to o)^\flat &= (o^a \to o). \end{aligned}$$

Notice that $\texttt{rank}(A^\sharp) \leq 2$.

In the potentially infinite context

$$\{u_A{:}A^\sharp \mid A \in \mathbb{T}\}$$

define inductively for any type A terms $V_A : o{\to}A, U_A : A{\to}A^\flat$.

$$\begin{array}{rcl} U_o &=& \lambda x{:}o.x;\\ V_o &=& \lambda x{:}o.x;\\ U_{A_1\to\ldots\to A_a\to o} &=& \lambda z{:}A\lambda x_1,\ldots,x_a{:}o.z(V_{A_1}x_1)\ldots(V_{A_a}x_a);\\ V_{A_1\to\ldots\to A_a\to o} &=& \lambda x{:}o\lambda y_1{:}A_1\ldots y_a{:}A_a.u_A x(U_{A_1}y_1)\ldots(U_{A_a}y_a), \end{array}$$

where $A = A_1{\to}\ldots{\to}A_a{\to}o$. Write $A_i = A_{i1}{\to}\ldots{\to}A_{in}{\to}o$.

Remark that for $C = A_1{\to}\ldots{\to}A_a{\to}B$ one has

$$U_C = \lambda z{:}C\lambda x_1,\ldots,x_a{:}o.U_B(z(V_{A_1}x_1)\ldots(V_{A_a}x_a)). \tag{1}$$

Indeed, both sides are equal to

$$\lambda z{:}A\lambda x_1,\ldots,x_a,y_1,\ldots,y_b{:}o.z(V_{A_1}x_1)\ldots(V_{A_a}x_a)(V_{B_1}y_1)\ldots(V_{B_b}y_b),$$

with $B = B_1{\to}\ldots{\to}B_b{\to}o$.

Notice that for a closed term M of type $A = A_1{\to}\ldots{\to}A_a{\to}o$ one can write

$$M = \lambda y_1{:}A_1\ldots y_a{:}A_a.y_i(M_1y_1\ldots y_a)\ldots(M_ny_1\ldots y_a).$$

Now verify that

$$\begin{array}{rcl} U_AM &=& \lambda x_1,\ldots,x_a{:}o.M(V_{A_1}x_1)\ldots(V_{A_a}x_a)\\ &=& \lambda\vec{x}.(V_{A_i}x_i)(M_1(V_{A_1}x_1)\ldots(V_{A_a}x_a))\ldots(M_n(V_{A_1}x_1)\ldots(V_{A_a}x_a))\\ &=& \lambda\vec{x}.u_{A_i}x_i(U_{A_{i1}}(M_1(V_{A_1}x_1)\ldots(V_{A_a}x_a)))\ldots(U_{A_{in}}(M_n(V_{A_1}x_1)\ldots(V_{A_a}x_a)))\\ &=& \lambda\vec{x}.u_{A_i}x_i(U_{B_1}M_1\vec{x})\ldots\ (U_{B_n}M_n\vec{x}), \end{array}$$

using (1), where $B_j = A_1{\to}\ldots{\to}A_a{\to}A_{ij}$ for $1 \le j \le n$ is the type of M_j. Hence we have that if $U_AM =_{\boldsymbol{\beta\eta}} U_AN$, then for $1 \le j \le n$

$$U_{B_j}M_j =_{\boldsymbol{\beta\eta}} U_{B_j}N_j.$$

Therefore it follows by induction on the complexity of M that if $U_AM =_{\boldsymbol{\beta\eta}} U_AN$, then $M =_{\boldsymbol{\beta\eta}} N$.

Now take as term for the reducibility $\Phi \equiv \lambda m{:}A\lambda u_{B_1}\ldots u_{B_k}.U_Am$, where the $\vec{u}$ are all the ones occurring in the construction of U_A. It follows that

$$A \le_{\beta\eta} B_1^\sharp{\to}\ldots{\to}B_k^\sharp{\to}A^\flat.$$

Since $\mathtt{rank}(B_1^\sharp{\to}\ldots{\to}B_k^\sharp{\to}A^\sharp) \le 3$, we are done. ■

For an alternative proof, see Exercise 3.6.9.

In the following proposition it will be proved that we can further reduce types to one particular type of rank 3. First do exercise 3.6.5 to get some intuition. We need the following notation.

3.4.3. NOTATION. (i) For $k \geq 0$ write

$$1_k = o^k \to o,$$

where in general $A^0 \to o = o$ and $A^{k+1} \to o = A \to (A^k \to o)$.

(ii) For $k_1, \ldots, k_n \geq 0$ write

$$(k_1, \ldots, k_n) = 1_{k_1} \to \ldots \to 1_{k_n} \to o.$$

(iii) For $k_{11}, \ldots, k_{1n_1}, \ldots, k_{m1}, \ldots, k_{mn_m} \geq 0$ write

$$\begin{pmatrix} k_{11} & \ldots & k_{1n_1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ k_{m1} & \ldots & k_{mn_m} \end{pmatrix} = (k_{11}, \ldots, k_{1n_1}) \to \ldots \to (k_{m1}, \ldots, k_{mn_m}) \to o.$$

Note the "matrix" has a dented right side (the n_i are unequal in general).

3.4.4. PROPOSITION. *Every type A of rank ≤ 3 is reducible to*

$$1_2 \to 1 \to 1 \to 2 \to o.$$

PROOF. Let A be a type of rank ≤ 3. It is not difficult to see that A is of the form

$$A = \begin{pmatrix} k_{11} & \ldots & k_{1n_1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ k_{m1} & \ldots & k_{mn_m} \end{pmatrix}$$

We will first reduce A to type $3 = 2 \to o$ using a term Φ containing free variables of type $1_2, 1, 1$ respectively acting as a 'pairing'. Consider the context

$$\{p{:}1_2, p_1{:}1, p_2{:}1\}.$$

Consider the notion of reduction p defined by the contraction rules

$$p_i(pM_1M_2) \to_p M_1.$$

[There now is a choice how to proceed: if you like syntax, then proceed; if you prefer models omit paragraphs starting with ♣ and jump to those starting with ♠.]

♣ This notion of reduction satisfies the subject reduction property. Moreover $\beta\eta p$ is Church-Rosser, see Pottinger [1981]. This can be used later in the proof. [Extending the notion of reduction by adding

$$p(p_1M)(p_2M) \to_s M$$

preserves the CR property. In the untyped calculus this is not the case, see Klop [1980] or Barendregt [1984], ch. 14.] Goto ♠.

♠ Given the pairing p, p_1, p_2 one can extend it as follows. Write

$$\begin{array}{rcll} p^1 &=& \lambda x{:}o.x; & \\ p^{k+1} &=& \lambda x_1 \dots x_n x_{n+1}{:}o.p(p^k x_1 \dots x_n)x_{n+1}; & \\ p^1_1 &=& \lambda x{:}o.x; & \\ p^{k+1}_{k+1} &=& p_2; & \\ p^{k+1}_i &=& \lambda z{:}o.p^k_i(p_1 z), & \text{for } i \leq k; \\ P^k &=& \lambda f_1 \dots f_k{:}1 \lambda z{:}o.p^k(f_1 z)\dots(f_k z); & \\ P^k_i &=& \lambda g{:}1 \lambda z{:}o.p^k_i(gz), & \text{for } i \leq k. \end{array}$$

We have that p^k acts as a coding for k-tuples of elements of type o with projections p^k_i. The P^k, P^k_i do the same for type 1. In context containing $\{f{:}1_k, g{:}1\}$ write

$$\begin{array}{rcl} f^{k\to 1} &=& \lambda z{:}o.f(p^k_1 z)\dots(p^k_k z); \\ g^{1\to k} &=& \lambda z_1 \dots z_k{:}o.f(p^k z_1 \dots z_k). \end{array}$$

Then $f^{k\to 1}$ is f moved to type 1 and $g^{1\to k}$ is g moved to type 1_k.

Using $\boldsymbol{\beta\eta}p$-convertibility one can show

$$\begin{array}{rcl} p^k_i(p^k z_1 \dots z_k) &=& z_i; \\ P^k_i(P^k f_1 \dots f_k) &=& f_i; \\ f^{k\to 1, 1\to k} &=& f. \end{array}$$

For $g^{1\to k, k\to 1} = g$ one needs s, the surjectivity of the pairing.

In order to define the term required for the reducibility start with the term $\Psi{:}A{\to}3$ (containing p, p_1, p_2 as only free variables). We need an auxiliary term Ψ^{-1}, acting as an inverse for Ψ in the presence of a "true pairing".

$$\begin{array}{rcl} \Psi &\equiv& \lambda M \lambda F{:}2.M \\ && [\lambda f_{11}{:}1_{k_{11}} \dots f_{1n_1}{:}1_{k_{1n_1}}.p_1(F(P^{n_1} f_{11}^{k_{11}\to 1} \dots f_{1n_1}^{k_{1n_1}\to 1})] \dots \\ && [\lambda f_{m1}{:}1_{k_{m1}} \dots f_{mn_m}{:}1_{k_{mn_m}}.p_m(F(P^{n_m} f_{m1}^{k_{m1}\to 1} \dots f_{mn_m}^{k_{mn_m}\to 1})]; \\ \Psi^{-1} &\equiv& \lambda N{:}(2{\to}o)\lambda K_1{:}(k_{11}, \dots, k_{1n_1}) \dots \lambda K_m{:}(k_{m1}, \dots, k_{mn_m}). \\ && N(\lambda f{:}1.p^m[K_1(P^{n_1}_1 f)^{1\to k_{11}} \dots (P^{n_1}_{n_1} f)^{1\to k_{1n_1}}] \dots \\ && \qquad [K_m(P^{n_m}_1 f)^{1\to k_{m1}} \dots (P^{n_m}_{n_m} f)^{1\to k_{1n_m}}]). \end{array}$$

Claim. For closed terms M_1, M_2 of type A we have

$$M_1 =_{\boldsymbol{\beta\eta}} M_2 \iff \Psi M_1 =_{\boldsymbol{\beta\eta}} \Psi M_2.$$

It then follows that for the reduction $A \leq_{\beta\eta} 1_2{\to}1{\to}1{\to}3$ we can take

$$\Phi = \lambda M{:}A.\lambda p{:}1_2 \lambda p_1, p_2{:}1.\Psi M.$$

It remains to show the claim. The only interesting direction is ($\Leftarrow$). This follows in two ways. We first show that

$$\Psi^{-1}(\Psi M) =_{\boldsymbol{\beta\eta}p} M. \tag{1}$$

We will write down the computation for the "matrix"

$$\begin{pmatrix} k_{11} & \\ k_{21} & k_{22} \end{pmatrix}$$

which is perfectly general.

$$\begin{array}{rll}
\Psi M & =_{\boldsymbol{\beta}} & \lambda F{:}2.M[\lambda f_{11}{:}1_{k_{11}}.p_1(F(P^1 f_{11}^{k_{11}\to 1}))] \\
& & [\lambda f_{21}{:}1_{k_{21}}\lambda f_{22}{:}1_{k_{22}}.p_2(F(P^2 f_{21}^{k_{21}\to 1} f_{22}^{k_{22}\to 1}))]; \\
\Psi^{-1}(\Psi M) & =_{\boldsymbol{\beta}} & \lambda K_1{:}(k_{11})\lambda K_2{:}(k_{21},k_{22}). \\
& & \Psi M(\lambda f{:}1.p^1[K_1(P_1^1 f)^{1\to k_{11}}[K_2(P_1^2 f)^{1\to k_{21}}(P_2^2 f)^{1\to k_{22}}]) \\
& \equiv & \lambda K_1{:}(k_1 1)\lambda K_2{:}(k_{21},k_{22}).\Psi M H, \text{ say,} \\
& =_{\boldsymbol{\beta}} & \lambda K_1 K_2.M[\lambda f_{11}.p_1(H(P^1 f_{11}^{k_{11}\to 1}))] \\
& & [\lambda f_{21}\lambda f_{22}.p_2(H(P^2 f_{21}^{k_{21}\to 1} f_{22}^{k_{22}\to 1}))]; \\
& =_{\boldsymbol{\beta}p} & \lambda K_1 K_2.M[\lambda f_{11}.p_1(p^2[K_1 f_{11}][..\text{`junk'}..])] \\
& & [\lambda f_{21}\lambda f_{22}.p_2(p^2[..\text{`junk'}..][K_2 f_{21} f_{22}])]; \\
& =_p & \lambda K_1 K_2.M(\lambda f_{11}.K_1 f_{11})(\lambda f_{21} f_{22}.K_2 f_{21} f_{22}) \\
& =_\eta & \lambda K_1 K_2.M K_1 K_2 \\
& =_\eta & M,
\end{array}$$

since

$$\begin{array}{rll}
H(P^1 f_{11}) & =_{\boldsymbol{\beta}p} & p^2[K_1 f_{11}][..\text{`junk'}..] \\
H(P^2 f_{21}^{k_{21}\to 1} f_{22}^{k_{22}\to 1}) & =_{\boldsymbol{\beta}p} & p^2[..\text{`junk'}..][K_2 f_{21} f_{22}].
\end{array}$$

The argument now can be finished in a model theoretic or syntactic way.

♣ If $\Psi M_1 =_{\boldsymbol{\beta\eta}} \Psi M_2$, then $\Psi^{-1}(\Psi M_1) =_{\boldsymbol{\beta\eta}} \Psi^{-1}(\Psi M_2)$. But then by (1) $M_1 =_{\boldsymbol{\beta\eta}p} M_2$. It follows from the Church-Rosser theorem for $\boldsymbol{\beta\eta}p$ that $M_1 =_{\boldsymbol{\beta\eta}} M_2$, since these terms do not contain p. Goto ■.

♠ If $\Psi M_1 =_{\boldsymbol{\beta\eta}} \Psi M_2$, then

$$\lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}(\Psi M_1) =_{\boldsymbol{\beta\eta}} \lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}(\Psi M_2).$$

Hence

$$\mathcal{M}(\omega) \models \lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}\Psi(M_1) = \lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}(\Psi M_2).$$

Let $\mathbf{q}$ be an actual pairing on ω with projections $\mathbf{q_1}, \mathbf{q_2}$. Then in $\mathcal{M}(\omega)$

$$(\lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}(\Psi M_1))\mathbf{q}\mathbf{q_1}\mathbf{q_2} = \lambda p{:}1_2\lambda p_1 p_2{:}1.\Psi^{-1}(\Psi M_2)\mathbf{q}\mathbf{q_1}\mathbf{q_2}.$$

Since $(\mathcal{M}(\omega), \mathbf{q}, \mathbf{q_1}, \mathbf{q_2})$ is a model of $\boldsymbol{\beta\eta}p$ conversion it follows from (1) that

$$\mathcal{M}(\omega) \models M_1 = M_2.$$

But then $M_1 =_{\boldsymbol{\beta\eta}} M_2$, by a result of Friedman [1975]. ■

We will see below, corollary 3.4.23 (i), that Friedman's result will follow from the reducibility theorem. Therefore the syntactic approach is preferable.

The proof of the next proposition is again syntactic. A warmup is exercise 3.6.7.

3.4.5. PROPOSITION. *Let A be a type of rank ≤ 2. Then*

$$2 \rightarrow A \leq_{\beta\eta} 1 \rightarrow 1 \rightarrow o \rightarrow A.$$

PROOF. Let $A \equiv (1^{k_1}, \ldots, 1^{k_n}) = 1_{k_1} \rightarrow \ldots 1_{k_n} \rightarrow o$. The term that will perform the reduction is relatively simple

$$\Phi \equiv \lambda M{:}(2 \rightarrow A)\lambda f, g{:}1\lambda z{:}o\lambda b_1{:}1_{k_1} \ldots \lambda b_n{:}1_{k_n}.M(\lambda h{:}1.f(h(g(hz)))).$$

In order to show that for all $M_1, M_2{:}2 \rightarrow A$ one has

$$\Phi M_1 =_{\beta\eta} \Phi M_2 \ \Rightarrow\ M_1 =_{\beta\eta} M_2,$$

we may assume w.l.o.g. that $A = 1_2 \rightarrow o$. A typical element of $2 \rightarrow 1_2 \rightarrow o$ is

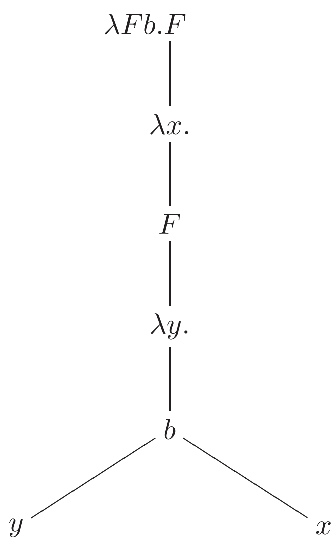
$$M \equiv \lambda F{:}2\lambda b{:}1_2.F(\lambda x.F(\lambda y.byx)).$$

Note that its translation has the following long $\boldsymbol{\beta\eta}$-nf

$$\begin{aligned} \Phi M &= \lambda f, g{:}1\lambda z{:}o\lambda b{:}1_2.f(N_x[x{:}= g(N_x[x{:}= z]])), \\ &\quad\ \text{where } N_x \equiv f(b(g(bzx))x), \\ &\equiv \lambda f, g{:}1\lambda z{:}o\lambda b{:}1_2.f(f(b(g(bz[g(f(b(g(bzz))z))]))[g(f(b(g(bzz))z))])). \end{aligned}$$

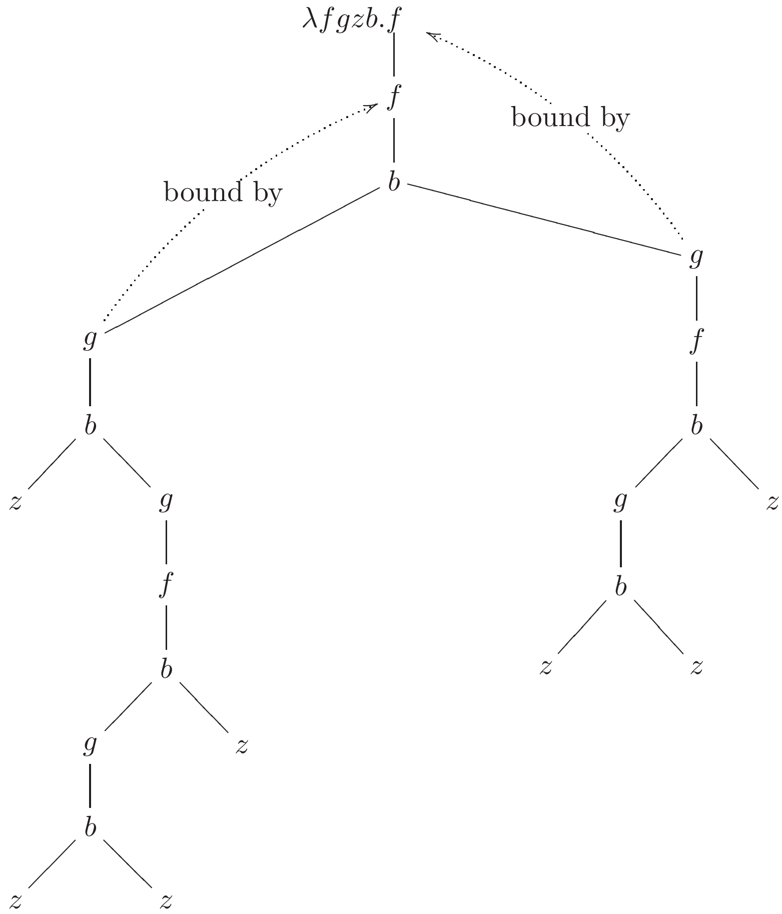
This term M and its translation have the following trees.

BT(M)



and

BT(ΦM)



Note that if we can 'read back' M from its translation ΦM, then we are done. Let $\mathtt{Cut}_{g\to z}$ be a syntactic operation on terms that replaces maximal subterms of the form gP by z. For example (omitting the abstraction prefix)

$$\mathtt{Cut}_{g\to z}(\Phi M) = f(f(bzz)).$$

Note that this gives us back the 'skeleton' of the term M, by reading f as $F(\lambda\odot$. The remaining problem is how to reconstruct the binding effect of each occurrence of the $\lambda\odot$. Using the idea of counting upwards lambda's, see de Bruijn [1972], this is accomplished by a realizing that the occurrence z coming from $g(P)$ should be bound at the position f just above where $\mathtt{Cut}_{g\to z}(P)$ matches in $\mathtt{Cut}_{g\to z}(\Phi M)$ above that z. For a precise inductive argument for this fact, see Statman [1980a], Lemma 5, or do exercise 3.6.10. ■

The following simple proposition brings almost to an end the chain of reducibility of types.

3.4.6. PROPOSITION.

$$1^4{\to}1_2{\to}o{\to}o \leq_{\beta\eta} 1_2{\to}o{\to}o.$$

PROOF. As it is equally simple, let us prove instead

$$1{\to}1_2{\to}o{\to}o \leq_{\beta\eta} 1_2{\to}o{\to}o.$$

Define $\Phi : (1{\to}1_2{\to}o{\to}o){\to}1_2{\to}o{\to}o$ by

$$\Phi \equiv \lambda M{:}(1{\to}1_2{\to}o{\to}o)\lambda b{:}1_2\lambda c{:}o.\lambda f{:}1\lambda b{:}1_2\lambda c{:}o.M(f^+)(b^+)c,$$

where

$$\begin{aligned} f^+ &= \lambda t{:}o.b(\#f)t; \\ b^+ &= \lambda t_1, t_2{:}o.b(\#b)(bt_1t_2); \\ \#f &= bcc; \\ \#b &= bc(bcc). \end{aligned}$$

The terms $\#f, \#b$ serve as recognizers ("Gödel numbers"). Notice that M of type $1{\to}1_2{\to}o{\to}o$ has a closed long $\boldsymbol{\beta\eta}$-nf of the form

$$M^{\rm nf} \equiv \lambda f{:}1\lambda b{:}1_2\lambda c{:}o.t$$

with t an element of the set T generated by the grammar

$$T{::} = c \mid fT \mid b\,T\,T.$$

Then for such M one has $\Phi M =_{\beta\eta} \Phi(M^{\rm nf}) \equiv M^+$ with

$$M^+ \equiv \lambda f{:}1\lambda b{:}1_2\lambda c{:}o.t^+,$$

where t^+ is inductively defined by

$$\begin{aligned} c^+ &= c; \\ (ft)^+ &= b(\#f)t^+; \\ (bt_1t_2)^+ &= b(\#b)(bt_1^+t_2^+). \end{aligned}$$

It is clear that $M^{\rm nf}$ can be constructed back from M^+. Therefore

$$\begin{aligned} \Phi M_1 =_{\beta\eta} \Phi M_2 \quad &\Rightarrow \quad M_1^+ =_{\beta\eta} M_2^+ \\ &\Rightarrow \quad M_1^+ \equiv M_2^+ \\ &\Rightarrow \quad M_1^{\rm nf} \equiv M_2^{\rm nf} \\ &\Rightarrow \quad M_1 =_{\beta\eta} M_2. \blacksquare \end{aligned}$$

By the same method one can show that any type of rank ≤ 2 is reducible to $\top$, do exercise 3.6.12

Combining propositions 3.4.2-3.4.6 we can complete the proof of the reducibility theorem.

3.4.7. Theorem (Reducibility theorem, Statman [1980a]). *Let*

$$\top = 1_2 \to o \to o.$$

Then

$$\forall A \in \mathbb{T}\ A \leq_{\beta\eta} \top.$$

Proof. Let A be any type. Harvesting the results we obtain

$$\begin{array}{rlll} A & \leq_{\beta\eta} & B, & \text{with } \mathtt{rank}(B) \leq 3, \text{ by 3.4.2,} \\ & \leq_{\beta\eta} & 1_2 \to 1^2 \to 2 \to o, & \text{by 3.4.4,} \\ & \leq_{\beta\eta} & 2 \to 1_2 \to 1^2 \to o, & \text{by simply permuting arguments,} \\ & \leq_{\beta\eta} & 1^2 \to o \to 1_2 \to 1^2 \to o, & \text{by 3.4.5,} \\ & \leq_{\beta\eta} & 1_2 \to o \to o, & \text{by an other permutation and 3.4.6} \blacksquare \end{array}$$

Now we turn attention to the type hierarchy, Statman [1980a].

3.4.8. Definition. For the ordinals $\alpha \leq \omega + 3$ define the type $A_\alpha \in \mathbb{T}(\lambda^o_\to)$ as follows.

$$\begin{array}{rcl} A_0 & = & o; \\ A_1 & = & o \to o; \\ & \dots & \\ A_k & = & o^k \to o; \\ & \dots & \\ A_\omega & = & 1 \to o \to o; \\ A_{\omega+1} & = & 1 \to 1 \to o \to o; \\ A_{\omega+2} & = & 3 \to o \to o; \\ A_{\omega+3} & = & 1_2 \to o \to o. \end{array}$$

3.4.9. Proposition. *For* $\alpha, \boldsymbol{\beta} \leq \omega + 3$ *one has*

$$\alpha \leq \boldsymbol{\beta} \Rightarrow A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}}.$$

Proof. For all finite k one has $A_k \leq_{\beta\eta} A_{k+1}$ via the map

$$\Phi_{k,k+1} \equiv \lambda m{:}A_k \lambda z x_1 \dots x_k{:}o.m x_1 \dots x_k =_{\beta\eta} \lambda m{:}A_k.\mathsf{K}m.$$

Moreover, $A_k \leq_{\beta\eta} A_\omega$ via

$$\Phi_{k,\omega} \equiv \lambda m{:}A_k \lambda f{:}1 \lambda x{:}o.m(\mathbf{c}_1 f x) \dots (\mathbf{c}_k f x).$$

Then $A_\omega \leq_{\beta\eta} A_{\omega+1}$ via

$$\Phi_{\omega,\omega+1} \equiv \lambda m{:}A_\omega \lambda f, g{:}1 \lambda x{:}o.m f x.$$

Now $A_{\omega+1} \leq_{\beta\eta} A_{\omega+2}$ via

$$\Phi_{\omega+1,\,\omega+2} \equiv \lambda m{:}A_{\omega+1}\lambda H{:}3\lambda x{:}o.H(\lambda f{:}1.H(\lambda g{:}1.mfgx)).$$

Finally, $A_{\omega+2} \leq_{\beta\eta} A_{\omega+3} = \top$ because of the reducibility theorem 3.4.7. See also exercise 4.1.9 for a concrete term $\Phi_{\omega+2,\,\omega+3}$. ■

3.4.10. Proposition. *For $\alpha, \boldsymbol{\beta} \leq \omega+3$ one has*

$$\alpha \leq \boldsymbol{\beta} \Leftarrow A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}}.$$

Proof. This will be proved in 3.5.32. ■

3.4.11. Corollary. *For $\alpha, \boldsymbol{\beta} \leq \omega+3$ one has*

$$A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}} \iff \alpha \leq \boldsymbol{\beta}.$$

For a proof that these types $\{A_\alpha\}_{\alpha\leq\omega+3}$ are a good representation of the reducibility classes we need some syntactic notions.

3.4.12. Definition. A type $A \in \mathbb{T}(\lambda^o_\to)$ is called *large* if it has a negative subterm occurrence of the form $B_1\to\ldots\to B_n\to o$, with $n \geq 2$; A is *small* otherwise.

3.4.13. Example. $1_2\to o\to o$ is large; $(1_2\to o)\to o$ and $3\to o\to o$ are small.

Now we will partition the types $\mathbb{T} = \mathbb{T}(\lambda^o_\to)$ in the following classes.

3.4.14. Definition. Define the following sets of types.

$$\begin{array}{lcl}
\mathbb{T}_{-1} & = & \{A \mid A \text{ has no closed inhabitant}\};\\
\mathbb{T}_0 & = & \{o\to o\};\\
\mathbb{T}_1 & = & \{o^k\to o \mid k>1\};\\
\mathbb{T}_2 & = & \{1\to o^q\to o \mid q>0\} \cup \{(1_p\to o)\to o^q\to o \mid p>0, q\geq 0\};\\
\mathbb{T}_3 & = & \{A \mid A \text{ is small}, \mathtt{rank}(A)\in\{2,3\} \text{ and } A \notin \mathbb{T}_2\};\\
\mathbb{T}_4 & = & \{A \mid A \text{ is small and } \mathtt{rank}(A)>3\};\\
\mathbb{T}_5 & = & \{A \mid A \text{ is large}\}.
\end{array}$$

It is clear that the $\mathbb{T}_i$ form a partition of $\mathbb{T}$. A typical element of $\mathbb{T}_{-1}$ is o. This class we will not consider much.

3.4.15. Theorem (Hierarchy theorem, Statman [1980a]).

$$\begin{array}{lcll}
A\in\mathbb{T}_5 & \iff & A \sim_{\boldsymbol{\beta\eta}} 1_2\to o\to o; & \\
A\in\mathbb{T}_4 & \iff & A \sim_{\boldsymbol{\beta\eta}} 3\to o\to o; & \\
A\in\mathbb{T}_3 & \iff & A \sim_{\boldsymbol{\beta\eta}} 1\to 1\to o\to o; & \\
A\in\mathbb{T}_2 & \iff & A \sim_{\boldsymbol{\beta\eta}} 1\to o\to o; & \\
A\in\mathbb{T}_1 & \iff & A \sim_{\boldsymbol{\beta\eta}} o^k\to o, & \textit{for some } k>1;\\
A\in\mathbb{T}_0 & \iff & A \sim_{\boldsymbol{\beta\eta}} o\to o; & \\
A\in\mathbb{T}_{-1} & \iff & A \sim_{\boldsymbol{\beta\eta}} o. &
\end{array}$$

Proof. Since the $\mathbb{T}_i$ form a partition, it is sufficient to show just the $\Rightarrow$'s.

As to $\mathbb{T}_5$, it is enough to show that $1_2 \to o \to o \leq_{\beta\eta} A$, for every large type A, since we know already the converse. For this see Statman [1980a], lemma 7. As a warmup exercise do 3.6.20.

As to $\mathbb{T}_4$, it is shown in Statman [1980a], proposition 2, that if A is small, then $A \leq_{\beta\eta} 3 \to o \to o$. It remains to show that for any small type A of rank > 3 one has $3 \to o \to o \leq_{\beta\eta} A$. Do exercise 3.6.27.

As to $\mathbb{T}_3$, the implication is shown in Statman [1980a], lemma 12. The condition about the type in that lemma is equivalent to belonging to $\mathbb{T}_3$.

As to $\mathbb{T}_2$, do exercise 3.6.22(ii).

As to $\mathbb{T}_i$, with $i = 1, 0, -1$, notice that $\Lambda^{\emptyset}(o^k \to o)$ contains exactly k closed terms for $k \geq 0$. This is sufficient. ■

For an application in the next section we need a refinement of the hierarchy theorem.

3.4.16. Definition. Let A, B be types.

(i) A is *head-redicible to* B, notation $A \leq_h B$, iff for some closed term $\Phi \in \Lambda^{\emptyset}(A \to B)$ one has

$$\forall M_1, M_2{:}A\, [M_1 =_{\beta\eta} M_2 \iff \Phi M_1 =_{\beta\eta} \Phi M_2],$$

and moreover Φ is of the form

$$\Phi = \lambda m{:}A \lambda x_1 \dots x_b{:}B.m P_1 \dots P_a, \qquad (+)$$

with $m \notin \mathrm{FV}(P_1, \dots, P_a)$.

(ii) A is *multi head-reducible to* B, notation $A \leq_{h^+} B$, iff there are closed terms $\Phi_1, \dots, \Phi_m \in \Lambda^{\emptyset}(A \to B)$ each of the form (+) such that

$$\forall M_1, M_2{:}A\, [M_1 =_{\beta\eta} M_2 \iff \Phi_1 M_1 =_{\beta\eta} \Phi_1 M_2 \,\&\, \dots \,\&\, \Phi_m M_1 =_{\beta\eta} \Phi_m M_2.$$

(iii) Write $A \sim_h B$ iff $A \leq_h B \leq_h A$ and similarly
$A \sim_{h^+} B$ iff $A \leq_{h^+} B \leq_{h^+} A$.

3.4.17. Proposition. (i) *$A \leq_h B \Rightarrow A \leq_{\beta\eta} B$.*

(ii) *Let $A, B \in \mathbb{T}_i$, with $i \neq 2$. Then $A \sim_h B$.*

(iii) *Let $A, B \in \mathbb{T}_2$. Then $A \sim_{h^+} B$.*

(iv) *$A \leq_{\beta\eta} B \Rightarrow A \leq_{h^+} B$.*

Proof. (i) Trivial.

(ii) Suppose $A \leq_{\beta\eta} B$. By inspection of the proof of the hierarchy theorem in all cases except for $A \in \mathbb{T}_2$ one obtains $A \leq_h B$. Do exercise 3.6.24.

(iii) In the exceptional case one obtains $A \leq_{h^+} B$, see exercise 3.6.23. ■

(iv) By (ii) and (iii), using the hierarchy theorem.

3.4.18. Corollary (Hierarchy theorem (revisited), Statman [1980b]).

$$\begin{array}{lcl} A \in \mathbb{T}_5 & \iff & A \sim_h 1_2 \to o \to o; \\ A \in \mathbb{T}_4 & \iff & A \sim_h 3 \to o \to o; \\ A \in \mathbb{T}_3 & \iff & A \sim_h 1 \to 1 \to o \to o; \\ A \in \mathbb{T}_2 & \iff & A \sim_{h^+} 1 \to o \to o; \\ A \in \mathbb{T}_1 & \iff & A \sim_{h^+} o^2 \to o; \\ A \in \mathbb{T}_0 & \iff & A \sim_h o \to o; \\ A \in \mathbb{T}_{-1} & \iff & A \sim_h o. \end{array}$$

Proof. The only extra fact to verify is that $o^k \to o \leq_{h^+} o^2 \to o$. ■

Applications of the reducibility theorem

The reducibility theorem has several consequences.

3.4.19. Definition. Let $\mathcal{C}$ be a class of $\lambda_\to$ models. $\mathcal{C}$ is called *complete* iff

$$\forall M, N \in \Lambda^\emptyset [\mathcal{C} \models M = N \iff M =_{\boldsymbol{\beta\eta}} N].$$

3.4.20. Definition. (i) $\mathcal{T} = \mathcal{T}_{b,c}$ is the algebraic structure of trees inductively defined as follows.

$$\mathcal{T} = c \mid b\ \mathcal{T}\ \mathcal{T}$$

(ii) For a $\lambda_\to$ model $\mathcal{M}$ we say that $\mathcal{T}$ *can be embedded into* $\mathcal{M}$, notation $\mathcal{T} \hookrightarrow M$, iff there exist $b_0 \in \mathcal{M}(o \to o \to o), c_0 \in \mathcal{M}(o)$ such that

$$\forall t, s \in \mathcal{T} [t \neq s \Rightarrow \mathcal{M} \models t^{\text{cl}} b_0 c_0 \neq s^{\text{cl}} b_0 c_0],$$

where $u^{\text{cl}} = \lambda b{:}o \to o \to o \lambda c{:}o.u$, is the closure of $u \in \mathcal{T}$.

The elements of $\mathcal{T}$ are binary trees with c on the leaves and b on the connecting nodes. Typical examples are $c, bcc, bc(bcc)$ and $b(bcc)c$. The existence of an embeding using b_0, c_0 implies for example that $b_0 c_0 (b_0 c_0 c_0), b_0 c_0 c_0$ and c_0 are mutually different in $\mathcal{M}$.

Note that $\mathcal{T} \not\hookrightarrow \mathcal{M}(2)$. To see this, write $gx = bxx$. One has $g^2(c) \neq g^4(c)$, but $\mathcal{M}(2) \models \forall g{:}o \to o \forall c{:}o. g^2(c) = g^4(c)$, do exercise 3.6.13.

3.4.21. Lemma. (i) $\Pi_{i \in I} \mathcal{M}_i \models M = N \iff \forall i \in I. \mathcal{M}_i \models M = N$.

(ii) $M \in \Lambda^\emptyset(\top) \iff \exists s \in \mathcal{T}. M =_{\boldsymbol{\beta\eta}} s^{\text{cl}}$.

Proof. (i) Since $[\![M]\!]^{\Pi_{i \in I} \mathcal{M}_i} = \boldsymbol{\lambda} i \in I. [\![M]\!]^{\mathcal{M}_i}$.

(ii) By an analysis of the possible shape of the normal forms of terms of type $\top = 1_2 \to o \to o$. ■

3.4.22. Theorem (1-section theorem, Statman [1985]). *$\mathcal{C}$ is complete iff there is an (at most countable) family $\{\mathcal{M}_i\}_{i \in I}$ of structurs in $\mathcal{C}$ such that*

$$\mathcal{T} \hookrightarrow \Pi_{i \in I} \mathcal{M}_i.$$

PROOF. ($\Rightarrow$) Suppose $\mathcal{C}$ is complete. Let $t, s \in \mathcal{T}$. Then

$$\begin{array}{rcll} t \neq s & \Rightarrow & t^{\mathrm{cl}} \neq_{\beta\eta} s^{\mathrm{cl}} & \\ & \Rightarrow & \mathcal{C} \not\models t^{\mathrm{cl}} = s^{\mathrm{cl}}, & \text{by completeness,} \\ & \Rightarrow & \mathcal{M}_{ts} \models t^{\mathrm{cl}} \neq s^{\mathrm{cl}}, & \text{for some } \mathcal{M}_{st} \in \mathcal{C}, \\ & \Rightarrow & \mathcal{M}_{ts} \models t^{\mathrm{cl}} b_{ts} c_{ts} \neq s^{\mathrm{cl}} b_{ts} c_{ts}, & \end{array}$$

for some $b_{ts} \in \mathcal{M}(o \to o \to o)$, $c_{ts} \in \mathcal{M}(o)$ by extensionality. Note that in the third implication the axiom of (countable) choice is used.

It now follows by lemma 3.4.21(i) that

$$\Pi_{t \neq s} \mathcal{M}_{ts} \models t^{\mathrm{cl}} \neq s^{\mathrm{cl}},$$

since they differ on the pair $b_0 c_0$ with $b_0(ts) = b_{ts}$ and similarly for c_0.

($\Leftarrow$) Suppose $\mathcal{T} \hookrightarrow \Pi_{i \in I} \mathcal{M}_i$ with $\mathcal{M}_i \in \mathcal{C}$. Let M, N be closed terms of some type A. By soundness one has

$$M =_{\beta\eta} N \;\Rightarrow\; \mathcal{C} \models M = N.$$

For the converse, let by the reducibility theorem $F : A \to \top$ be such that

$$M =_{\beta\eta} N \iff FM =_{\beta\eta} FN,$$

for all $M, N \in \Lambda^{\emptyset}$. Then

$$\begin{array}{rcll} \mathcal{C} \models M = N & \Rightarrow & \Pi_{i \in I} \mathcal{M}_i \models M = N, & \text{by the lemma,} \\ & \Rightarrow & \Pi_{i \in I} \mathcal{M}_i \models FM = FN, & \\ & \Rightarrow & \Pi_{i \in I} \mathcal{M}_i \models t^{\mathrm{cl}} = s^{\mathrm{cl}}, & \end{array}$$

where t, s are such that

$$FM =_{\beta\eta} t^{\mathrm{cl}}, FN =_{\beta\eta} s^{\mathrm{cl}}, \qquad (*)$$

noting that every closed term of type $\top$ is $\beta\eta$-convertible to some u^{cl} with $u \in \mathcal{T}$. Now the chain of arguments continues as follows

$$\begin{array}{cll} \Rightarrow & t \equiv s, & \text{by the embedding property,} \\ \Rightarrow & FM =_{\beta\eta} FN, & \text{by (*),} \\ \Rightarrow & M =_{\beta\eta} N, & \text{by reducibility.} \blacksquare \end{array}$$

3.4.23. COROLLARY. (i) [Friedman [1975]] $\{\mathcal{M}_{\mathbb{N}}\}$ is complete.

(ii) [Plotkin [1985?]] $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ is complete.

(iii) $\{\mathcal{M}_{\mathbb{N}_\perp}\}$ is complete.

(iv) $\{\mathcal{M}_D\}_{D \text{ a finite cpo}}$, is complete.

PROOF. Immediate from the theorem. ■

The completeness of the collection $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ essentially states that for every pair of terms M, N of a given type A there is a number $n = n_{M,N}$ such that $\mathcal{M}_n \models M = N \Rightarrow M =_{\beta\eta} N$. Actually one can do better, by showing that n only depends on M.

3.4.24. PROPOSITION (Finite completeness theorem, Statman [1982]). *For every type $A \in \mathbb{T}(\lambda^o_\rightarrow)$ and every closed term M of type A there is a number $n = n_M$ such that for all closed terms N of type A one has*

$$\mathcal{M}_n \models M = N \iff M =_{\beta\eta} N.$$

PROOF. By the reduction theorem 3.4.7 it suffices to show this for $A = \top$. Let M a closed term of type $\top$ be given. Each closed term N of type $\top$ has as long $\boldsymbol{\beta\eta}$-nf

$$N = \lambda b{:}1_2\lambda c{:}o.s_N,$$

where $s_N \in \mathcal{T}$. Let $\mathbf{p} : \mathbb{N}\rightarrow\mathbb{N}\rightarrow\mathbb{N}$ be an injective pairing on the integers such that $\mathbf{p}(k_1, k_2) > k_i$. Take

$$n_M = [\![M]\!]^{\mathcal{M}_\omega}\mathbf{p}\,0 + 1.$$

Define $\mathbf{p}'{:}X^2_{n+1}\rightarrow X_{n+1}$, where $X_{n+1} = \{0, \dots, n+1\}$, by

$$\begin{array}{rcll} \mathbf{p}'(k_1, k_2) & = & \mathbf{p}(k_1, k_2), & \text{if } k_1, k_2 \leq n\mathbf{p}(k_1, k_2) \leq n; \\ & = & n+1 & \text{else.} \end{array}$$

Suppose $\mathcal{M}_n \models M = N$. Then $[\![M]\!]^{\mathcal{M}_n}\mathbf{p}'\,0 = [\![N]\!]^{\mathcal{M}_n}\mathbf{p}'\,0$. By the choice of n it follows that $[\![M]\!]^{\mathcal{M}_n}\mathbf{p}\,0 = [\![N]\!]^{\mathcal{M}_n}\mathbf{p}\,0$ and hence $s_M = s_N$. Therefore $M =_{\beta\eta} N$. ∎

3.4.25. DEFINITION (Reducibility Hierarchy, Statman [1980a]). For the ordinals $\alpha \leq \omega + 3$ define the type $A_\alpha \in \mathbb{T}(\lambda^o_\rightarrow)$ as follows.

$$\begin{array}{rcl} A_0 & = & o; \\ A_1 & = & o\rightarrow o; \\ & \dots & \\ A_k & = & o^k\rightarrow o; \\ & \dots & \\ A_\omega & = & 1\rightarrow o\rightarrow o; \\ A_{\omega+1} & = & 1\rightarrow 1\rightarrow o; \\ A_{\omega+2} & = & 3\rightarrow o\rightarrow o; \\ A_{\omega+3} & = & 1_2\rightarrow o\rightarrow o. \end{array}$$

3.4.26. PROPOSITION. *For $\alpha, \boldsymbol{\beta} \leq \omega + 3$ one has*

$$\alpha \leq \boldsymbol{\beta} \Rightarrow A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}}.$$

PROOF. For all finite k one has $A_k \leq_{\beta\eta} A_{k+1}$ via the map

$$\Phi_{k,k+1} \equiv \lambda m{:}A_k \lambda z x_1 \dots x_k{:}o.m x_1 \dots x_k =_{\beta\eta} \lambda m{:}A_k.\mathsf{K}m.$$

Moreover, $A_k \leq_{\beta\eta} A_\omega$ via

$$\Phi_{k,\omega} \equiv \lambda m{:}A_k \lambda f{:}1 \lambda x{:}o.m(\mathbf{c}_1 f x) \dots (\mathbf{c}_k f x).$$

Then $A_\omega \leq_{\beta\eta} A_{\omega+1}$ via

$$\Phi_{\omega,\,\omega+1} \equiv \lambda m{:}A_\omega \lambda f, g{:}1 \lambda x{:}o.m f x.$$

Now $A_{\omega+1} \leq_{\beta\eta} A_{\omega+2}$ via

$$\Phi_{\omega+1,\,\omega+2} \equiv \lambda m{:}A_{\omega+1} \lambda H{:}3 \lambda x{:}o.H(\lambda f{:}1.H(\lambda g{:}1.m f g x)).$$

Finally, $A_{\omega+2} \leq_{\beta\eta} A_{\omega+3} = \top$ because of the reducibility theorem 3.4.7. See also exercise 4.1.9 for a concrete term $\Phi_{\omega+2,\,\omega+3}$. ■

3.4.27. PROPOSITION. *For $\alpha, \boldsymbol{\beta} \leq \omega + 3$ one has*

$$\alpha \leq \boldsymbol{\beta} \Leftarrow A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}}.$$

PROOF. This will be proved in 3.5.32. ■

3.4.28. COROLLARY. *For $\alpha, \boldsymbol{\beta} \leq \omega + 3$ one has*

$$A_\alpha \leq_{\beta\eta} A_{\boldsymbol{\beta}} \iff \alpha \leq \boldsymbol{\beta}.$$

3.5. The five canonical term-models

The open terms of $\lambda^o_\to$ form an extensional model, the term-model $\mathcal{M}_{\Lambda_o}$. One may wonder whether there are also closed term-models, like in the untyped lambda calculus. If no constants are present, then this is not the case, since there are e.g. no closed terms of ground type o. In the presence of constants matters change. We will first show how a set of constants $\mathcal{D}$ gives rise to an extensional equivalence relation on $\Lambda^\emptyset_o[\mathcal{D}]$, the set of closed terms with constants from $\mathcal{D}$. Then we define canonical sets of constants and prove that for these the resulting equivalence relation is also a congruence, i.e. determines a term-model. After that it will be shown that for all sets $\mathcal{D}$ of constants with enough closed terms the extensional equivalence determines a term-model. Up to elementary equivalence (satisfying the same set of equations between closed pure terms, i.e. closed terms without any constants) all models, for which the equality on type o coincides with $=_{\beta\eta}$, can be obtained in this way. From now on $\mathcal{D}$ will range over sets of constants such that there are closed terms for every type A (i.e. in $\Lambda^\emptyset_o[\mathcal{D}](A)$).

3.5.1. DEFINITION. Let $M, N \in \Lambda^\emptyset_o[\mathcal{D}](A)$ with $A = A_1 \to \dots \to A_a \to o$.