Five Easy Pieces

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Types $\mathbb{T} := 0 \mid \mathbb{T} \to \mathbb{T}$

Notation

$$\begin{array}{rcl} \mathbf{0} & \triangleq & 0 \\ \boldsymbol{n+1} & \triangleq & \boldsymbol{n}\to 0 \qquad \mathbf{1}_2 = 0\to 0\to 0 \end{array}$$

Examples

$\mathsf{T}_5 = \top$	$\triangleq$	$\mathbf{1}_2\to 0\to 0$	type of trees	$\mathcal{C}_5 = \{b^{1_2}, c^0\}$
$\mathsf{T}_4 = \mathsf{M}$	$\triangleq$	$\mathbf{3}\to 0\to 0$	'monster' type	$\mathcal{C}_4 = \{\Phi^3, c^0\}$
$\mathsf{T}_3 = \mathsf{W}$	$\triangleq$	$\mathbf{1}\to\mathbf{1}\to 0\to 0$	type of words $\{f, g\}^*$	$\mathcal{C}_3 = \{f^1, g^1, c^0\}$
$\mathsf{T}_2 = \mathsf{N}$	$\triangleq$	$\mathbf{1}\to 0\to 0$	type of Church numerals	$\mathcal{C}_2 = \{f^1, c^0\}$
$\mathsf{T}_1 = \mathsf{B}$	$\triangleq$	$0\to 0\to 0$	type of Booleans	$\mathcal{C}_1 = \{c^0, d^0\}$
[$\mathsf{T}_0 = \mathsf{S}$	$\triangleq$	$0\to 0$	singleton type	$\mathcal{C}_0 = \{c^0\}$]
[$\mathsf{T}_{-1} = \mathsf{E}$	$\triangleq$	0	empty type	$\mathcal{C}_{-1} = \emptyset$]

Constants are considered as variables that we promise not to bind

$\lambda_{\rightarrow}^{\text{Curry}}$

$$\Gamma, x{:}A \vdash x : A$$

$$\frac{\Gamma \vdash M : (A \rightarrow B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B} \qquad \frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A \rightarrow B)}$$

$\lambda_{\rightarrow}^{\text{Church}}$

$$\begin{array}{rcl} & & x^A \in \Lambda(A) \\ M \in \Lambda(A \rightarrow B),\ N \in \Lambda(A) & \Rightarrow & (MN) \in \Lambda(B) \\ M \in \Lambda(B) & \Rightarrow & (\lambda x^A.M) \in \Lambda(A \rightarrow B) \end{array}$$

$$\Lambda^{\emptyset}(B) \triangleq \{M \in \Lambda(B) \mid \mathrm{FV}(M) = \emptyset\}$$

$$\Lambda^{\emptyset}[x^{A_1}, \ldots, x_n^{A_n}](B) \triangleq \{M \in \Lambda(B) \mid \mathrm{FV}(M) \subseteq \{x^{A_1}, \ldots, x_n^{A_n}\}\}$$

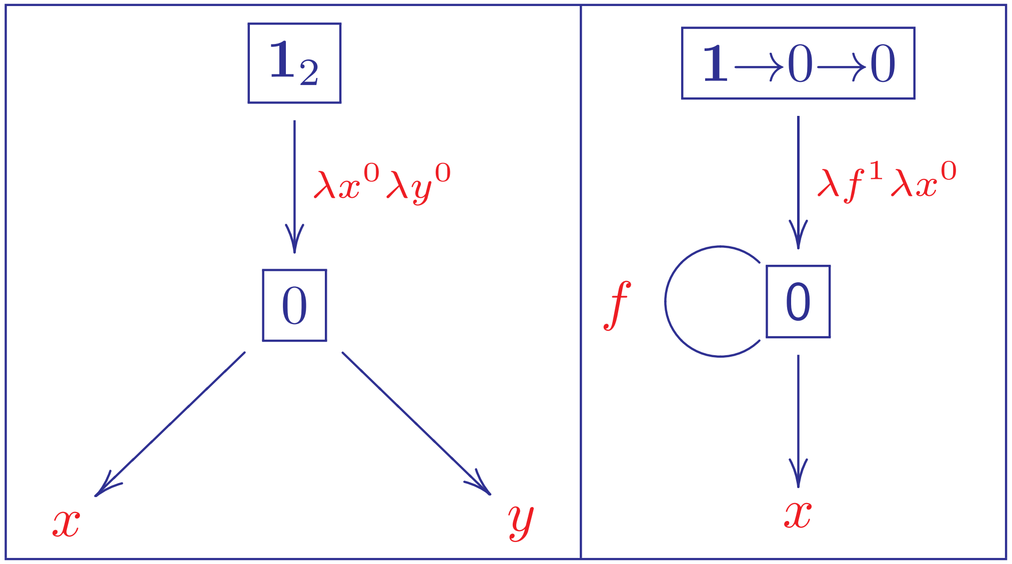
Fact. $\Lambda^{\emptyset}(A_1 \rightarrow \ldots A_n \rightarrow 0) \cong \{\lambda x^{A_1} \ldots x_n^{A_n}.M \mid M \in \Lambda^{\emptyset}[x^{A_1}, \ldots, x_n^{A_n}](0)\}$

For terms not containing λK-redexes the two versions are the same:

$x_1{:}A_1, \ldots, x_n{:}A_n \vdash M : B \iff \exists! M' \in \Lambda^{\emptyset}[x^{A_1}, \ldots, x_n^{A_n}](B).|M'| \equiv M$

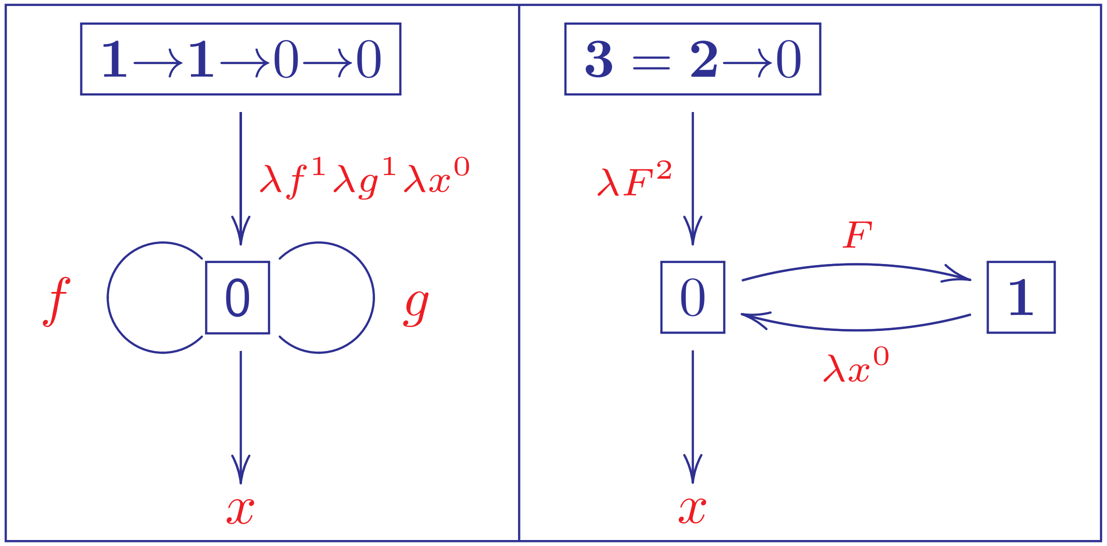
Difference: in the Church version one has to specify the type of I in Kx^AI

$$x : A \vdash \mathsf{K}x\mathsf{I} : A \qquad \text{vs} \qquad \mathsf{K}x^A\mathsf{I}^{B \rightarrow B} \in \Lambda^{\emptyset}[x^A](A)$$



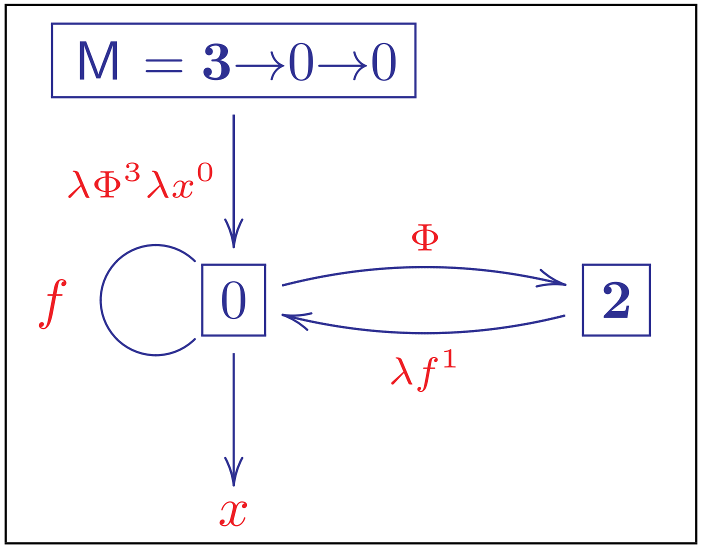
$$\Lambda^{\emptyset}(\mathbf{1}_2) = \{\lambda xy.x, \lambda xy.y\}$$

$$\Lambda^{\emptyset}(\mathbf{1}{\to}0{\to}0) = \{\lambda fx.f^n x \mid n \geq 0\}$$



$$\Lambda^{\emptyset}(\mathbf{1}{\to}\mathbf{1}{\to}0{\to}0) = \{\lambda fgx.w_{fg}x \mid w_{fg}{\in}\{f,g\}^*, \text{ application to the right}\}$$

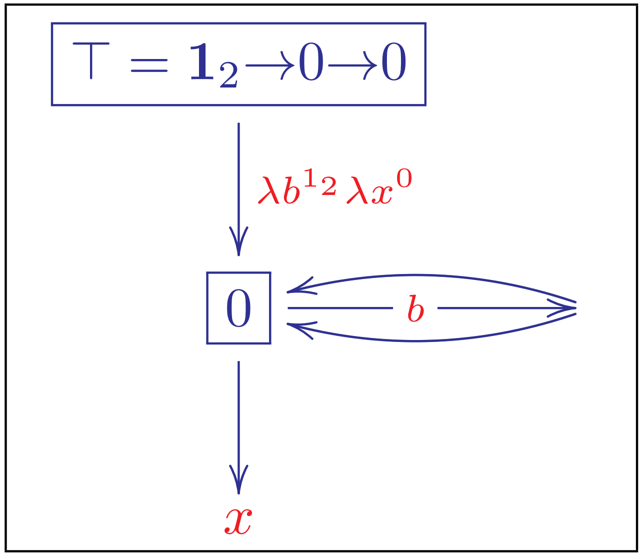
$$\Lambda^{\emptyset}(\mathbf{3}) = \{\lambda F.F(\lambda x_1.F(\lambda x_2.\cdots F(\lambda x_n.x_i)..)) \mid 1 \leq i \leq n\}$$



$\Lambda^{\emptyset}(\mathsf{M}) =$

$\{\lambda \Phi x.\Phi(\lambda f_1.w_{f_1}\Phi(\lambda f_2.w_{f_1 f_2}\ldots\Phi(\lambda f_n.w_{f_1\ldots f_n}x)..))$

$\mid w_{f_1\cdots f_k}\in\{f_1,\ldots,f_k\}^*,$ application to the right$\}$



$\Lambda^{\emptyset}(\top) = \{\lambda bc.t \mid t\in \mathsf{Tree}\}$

with Tree ::= $c \mid b$ Tree Tree

Definition. Let $A, B \in \mathbb{T}$. Then A is *$\boldsymbol{\beta\eta}$-reducible*

(*$\boldsymbol{\beta\eta}$-equivalent*, *strictly $\boldsymbol{\beta\eta}$-reducible*) to (with) B iff

$$A \leq_{\beta\eta} B \quad \stackrel{\triangle}{\iff} \quad \exists \Phi \in \Lambda^{\emptyset}(A \to B).[\forall M, N \in \Lambda^{\emptyset}(A).\Phi(M) =_{\boldsymbol{\beta\eta}} \Phi(N) \iff M =_{\boldsymbol{\beta\eta}} N]$$

$$A \sim_{\boldsymbol{\beta\eta}} B \quad \stackrel{\triangle}{\iff} \quad A \leq_{\boldsymbol{\beta\eta}} B \; \& \; B \leq_{\boldsymbol{\beta\eta}} A$$

$$A <_{\boldsymbol{\beta\eta}} B \quad \stackrel{\triangle}{\iff} \quad A \leq_{\boldsymbol{\beta\eta}} B \; \& \; B \not\leq_{\boldsymbol{\beta\eta}} A$$

Theorem. (i) One has $[1_3 \triangleq 0^3 \to 0 = 0 \to (0 \to (0 \to 0))]$

$$T_{-1} = 0 <_{\boldsymbol{\beta\eta}} T_0 = 1 <_{\boldsymbol{\beta\eta}} T_1 = 1_2 <_{\boldsymbol{\beta\eta}} 1_3 <_{\boldsymbol{\beta\eta}} \ldots <_{\boldsymbol{\beta\eta}} T_2 <_{\boldsymbol{\beta\eta}} T_3 <_{\boldsymbol{\beta\eta}} T_4 <_{\boldsymbol{\beta\eta}} T_5$$

(ii) Moreover, every $A \in \mathbb{T}$ is $\boldsymbol{\beta\eta}$-equivalent to one of these types

The definitions of $\leq_{\beta\eta}, \leq_h, \leq_{h^+}$ are due to Statman and he proved this theorem except for $T_3 \not\leq_{\beta\eta} T_4$, which was proved by Wil Dekkers

Definition. Let $A,\ B \equiv B_1 \to \ldots \to B_b \to 0 \in \mathbb{T}$.

(i) A is *head-reducible to* B, denoted by $A \leq_h B$, iff for some $\Phi \in \Lambda^{\emptyset}(A \to B)$

$$\forall M_1, M_2 \in \Lambda^{\emptyset}(A)\ [\Phi M_1 =_{\beta\eta} \Phi M_2 \iff M_1 =_{\beta\eta} M_2],$$

and moreover $\Phi \equiv \lambda m{:}A \lambda x_1{:}B_0 \ldots x_b{:}B_b.m P_1 \ldots P_a$, a 'Böhm transformation'(1) with $\mathrm{FV}(P_1, \ldots, P_a) \subseteq \{x_1, \ldots, x_b\}$ and $m \notin \{x_1 \ldots x_b\}$.

(ii) A is *multi head-reducible to* B, denotated by $A \leq_{h^+} B$, iff for some $\Phi_1, \ldots, \Phi_m \in \Lambda^{\emptyset}(A \to B)$, each of the form (1),

$$\forall M_1, M_2 \in \Lambda^{\emptyset}(A)\ [\Phi_1 M_1 =_{\beta\eta} \Phi_1 M_2\ \&\ldots\&\ \Phi_m M_1 =_{\beta\eta} \Phi_m \iff M_2 M_1 =_{\beta\eta} M_2]$$

This means that $\vec{\Phi}\colon \Lambda^{\emptyset}(A) \to (\Lambda^{\emptyset}(B))^m$ is injective.

(iii) Define $A \sim_h B,\ A <_h B$ and $A \sim_{h^+} B,\ A <_{h+} B$ as expected

Proposition. (i) $A \leq_h B \Rightarrow A \leq_{\beta\eta} B,\ A \leq_{\beta\eta} B \Rightarrow A \leq_{h^+} B$

(ii) $\mathbf{1}_2 \sim_{h^+} \mathbf{1}_3 \sim_{h^+} \ldots$, but $\mathbf{1}_3 \not\leq_{\beta\eta} \mathbf{1}_2$

(iii) $\mathbf{3} \leq_{\beta\eta} \mathbf{1} \to 0 \to 0$, but $\mathbf{3} \not\leq_h \mathbf{1} \to 0 \to 0$ (Bram Westerbaan)

Theorem. (i) $T_{-1} = 0 <_{h^+} T_0 = 1 <_{h^+} T_1 = 1_2 <_{h^+} T_2 <_{h^+} T_3 <_{h^+} T_4 <_{h^+} T_5$

(ii) Every $A \in \mathbb{T}$ is $\sim_{h^+}$ with one of these

Definition. Let $\mathcal{D}$ be a set of constants with various types in $\mathbb{T}$.

Let $M, N \in \Lambda^{\emptyset}[\mathcal{D}](A)$. $\mathcal{D}$-*equivalence* is defined by

$$M \approx_{\mathcal{D}} N \iff \forall\, F \in \Lambda^{\emptyset}[\mathcal{D}](A \to 0)\; FM =_{\beta\eta} FN$$

Note that for $M, N \in \Lambda^{\emptyset}[\mathcal{D}](A)$ and $F \in \Lambda^{\emptyset}[\mathcal{D}](A \to B)$ one has

$$M \approx_{\mathcal{D}} N \Rightarrow FM \approx_{\mathcal{D}} FN$$

Theorem. Let $F, G \in \Lambda^{\emptyset}[\mathcal{D}](A \to B)$. Then

$$F \approx_{\mathcal{D}} G \iff \forall M \in \Lambda^{\emptyset}[\mathcal{D}](A). FM \approx_{\mathcal{D}} GM$$

This is first proved for $\mathcal{C}_1, \ldots, \mathcal{C}_5$ and then for all $\mathcal{D}$ using type reducibility

Corollary. $\mathcal{M}(\mathcal{D}) \triangleq \Lambda^{\emptyset}[\mathcal{D}]/{\approx_{\mathcal{D}}}$ is an extensional term model

Theorem. The only such models are $\mathcal{M}(\mathcal{C}_1)$, $\mathcal{M}(\mathcal{C}_2), \ldots, \mathcal{M}(\mathcal{C}_5)$
apart from the trivial $\mathcal{M}(\mathcal{C}_0)$

Definition. $\mathsf{Th}(\mathcal{M}) \triangleq \{M = N \mid \exists A \in \mathbb{T}. M, N \in \Lambda^{\emptyset}(A) \;\&\; \mathcal{M} \models M = N\}$

Theorem. $\mathsf{Th}(\mathcal{M}(\mathcal{C}_5)), \mathsf{Th}(\mathcal{M}(\mathcal{C}_1)), \mathsf{Th}(\mathcal{M}(\mathcal{C}_0))$ are decidable.

In fact for all $A \in \mathbb{T}$ and $M, N \in \Lambda^{\emptyset}(A)$

$$\mathcal{M}(\mathcal{C}_5) \models M = N \iff M =_{\beta\eta} N$$

$$\mathcal{M}(\mathcal{C}_1) \models M = N \iff \lambda_{\rightarrow} + M = N \text{ is consistent,}$$

$\Leftrightarrow \forall F.[FM = c^0 \Leftrightarrow FN = c^0]$, which is decidable (V. Padovani)

$$\mathcal{M}(\mathcal{C}_0) \models M = N \iff 1 + 1 = 2$$

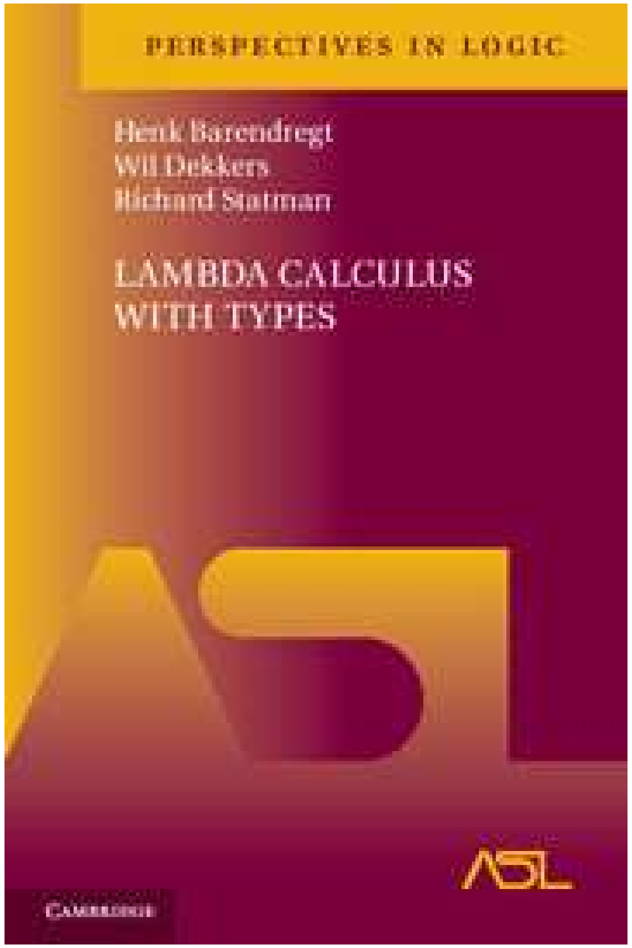
Differentiating equalities

$$\begin{array}{lrcl}
\mathcal{M}(\mathcal{C}_0) \models & \lambda f^1 x^0 . x & = & \lambda f^1 x^0 . f x \;\; (\ulcorner 0 \urcorner = \ulcorner 1 \urcorner) \\
\mathcal{M}(\mathcal{C}_1) \models & \lambda f^1 x^0 . f x & = & \lambda f^1 x^0 . f^2 x \;\; (\ulcorner 1 \urcorner = \ulcorner 2 \urcorner) \\
\mathcal{M}(\mathcal{C}_2) \models & \lambda f^1 g^1 x^0 . f(g(g(fx))) & = & \lambda f^1 g^1 x^0 . f(g(f(gx))) \\
\mathcal{M}(\mathcal{C}_3) \models & \lambda F^3 x^0 . F(\lambda f^1 . f(F(\lambda g^1 . g(fx)))) & = & \lambda F^3 x^0 . F(\lambda f^1 . f(F(\lambda g^1 . g(gx)))) \\
\mathcal{M}(\mathcal{C}_4) \models & \lambda h^{1_2} x^0 . h(hx(hxx))(hxx) & = & \lambda h^{1_2} x^0 . h(hxx)(h(hxx)x) \\
\mathcal{M}(\mathcal{C}_5) \models & \mathsf{I} c^0 & = & c^0
\end{array}$$

Open problems. (i) Are $\mathsf{Th}(\mathcal{M}(C_4)), \mathsf{Th}(\mathcal{M}(C_3)), \mathsf{Th}(\mathcal{M}(C_2))$ decidable?

(ii) Study reducibility in case there are more type atoms.

New book for P.-L.C: Happy Birthday



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