Regular Languages & Finite Automata

Now we are going to play a different 'ball-game'. Let $\Sigma = \{a, b\}$

S	$\to$	O | E
E	$\to$	λ | aEa | bEb
O	$\to$	a | b | aOa | bOb

G_1 a *context-free grammar*

Productions (always start with S)

S $\Rightarrow$ E $\Rightarrow$ aEa $\Rightarrow$ abEba $\Rightarrow$ abba

S $\Rightarrow$ E $\Rightarrow$ bEb $\Rightarrow$ baEab $\Rightarrow$ babEbab $\Rightarrow$ babaEabab $\Rightarrow$ babaabab

S $\Rightarrow$ O $\Rightarrow$ bOb $\Rightarrow$ bab

S $\Rightarrow$ O $\Rightarrow$ bOb $\Rightarrow$ baOab $\Rightarrow$ babab

$L(G) = \{w \in \Sigma^* \mid w$ is a palindrome$\}$,

where w is a palindrome if $w = w^R$

and word reversal is defined by $\lambda^R = \lambda$, $(w\sigma)^R = \sigma(w^R)$ for $\sigma \in \Sigma^*$

$$S \rightarrow \lambda \mid aSb \qquad G_2$$

All possible productions

$$\begin{array}{ccccccccccccc}
S & \Longrightarrow & aSb & \Longrightarrow & aaSbb & \Longrightarrow & aaaSbbb & \Longrightarrow & \cdots & \Longrightarrow & a^nSb^n & \Longrightarrow & \cdots \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \vdots & & \Downarrow & & \\
\lambda & & ab & & aabb & & aaabbb & & \cdots & & a^nb^n & & \cdots
\end{array}$$

Therefore

$$L(G_2) = \{a^nb^n \mid n \geq 0\}$$

Another grammar

$$S \rightarrow ab \mid aSb \qquad G_3$$

$$L(G_3) = \{a^nb^n \mid n > 0\}$$

Definition. A language L is called *context-free*

if for some context free grammar G one has $L = L(G)$

Let $\Sigma^* = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$

Claim. $L = \{\mathtt{a}^n \mathtt{b}^m \mathtt{c}^{2n+1} \mid n \geq 0\}$ is context-free

Let us first show

$L' = \{\mathtt{a}^n \mathtt{c}^{2n+1} \mid n \geq 0\}$ is context-free

Use G' given by

S	$\rightarrow$	c | aScc

For L use G given by

S	$\rightarrow$	Bc | aScc
B	$\rightarrow$	λ | bB

Fact. $\{\mathtt{a}^n \mathtt{b}^n \mathtt{c}^n \mid n \geq 0\}$ is *not* context-free

Formal definition of a context-free grammar

Let Σ be a finite alphabet

A *context-free grammar* G over Σ needs a finite set V of *auxiliary* symbols

and consists of productionrules of the vorm

$$X \to w$$

with $X \in V$ and $w \in (\Sigma \cup V)^*$. There is an $S \in V$ (start)

Using G a language is generated using a relation $\Rightarrow$ ('*produces*') defined as follows

(where u, v, w, x, y are arbitrary elements of $(\Sigma \cup V)^*$)

$$\begin{array}{rcl} X \to w & \text{implies} & xXy \Rightarrow xwy \\ u \Rightarrow v,\ v \Rightarrow w & \text{implies} & u \Rightarrow w \end{array}$$

The *language generated by G* is

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow w\}$$

Notation. $X \to w_1 \mid w_2$ abbreviates $\begin{array}{ccc} X & \to & w_1 \\ X & \to & w_2 \end{array}$

A language L is *context-free* if $L = L(G)$ for some context-free G

The *context sensitive languages* start with production rules of the form

$$uXv \to uwv,$$

with $u, v \in (\Sigma \cup V)^*$ arbitrary and $w \in \Sigma^*$ not λ.

For the *enumerable languages* production rules are of the form

$$uXv \to uwv,$$

with $u, v \in (\Sigma \cup V)^*$ and $w \in \Sigma^*$ arbitrary.

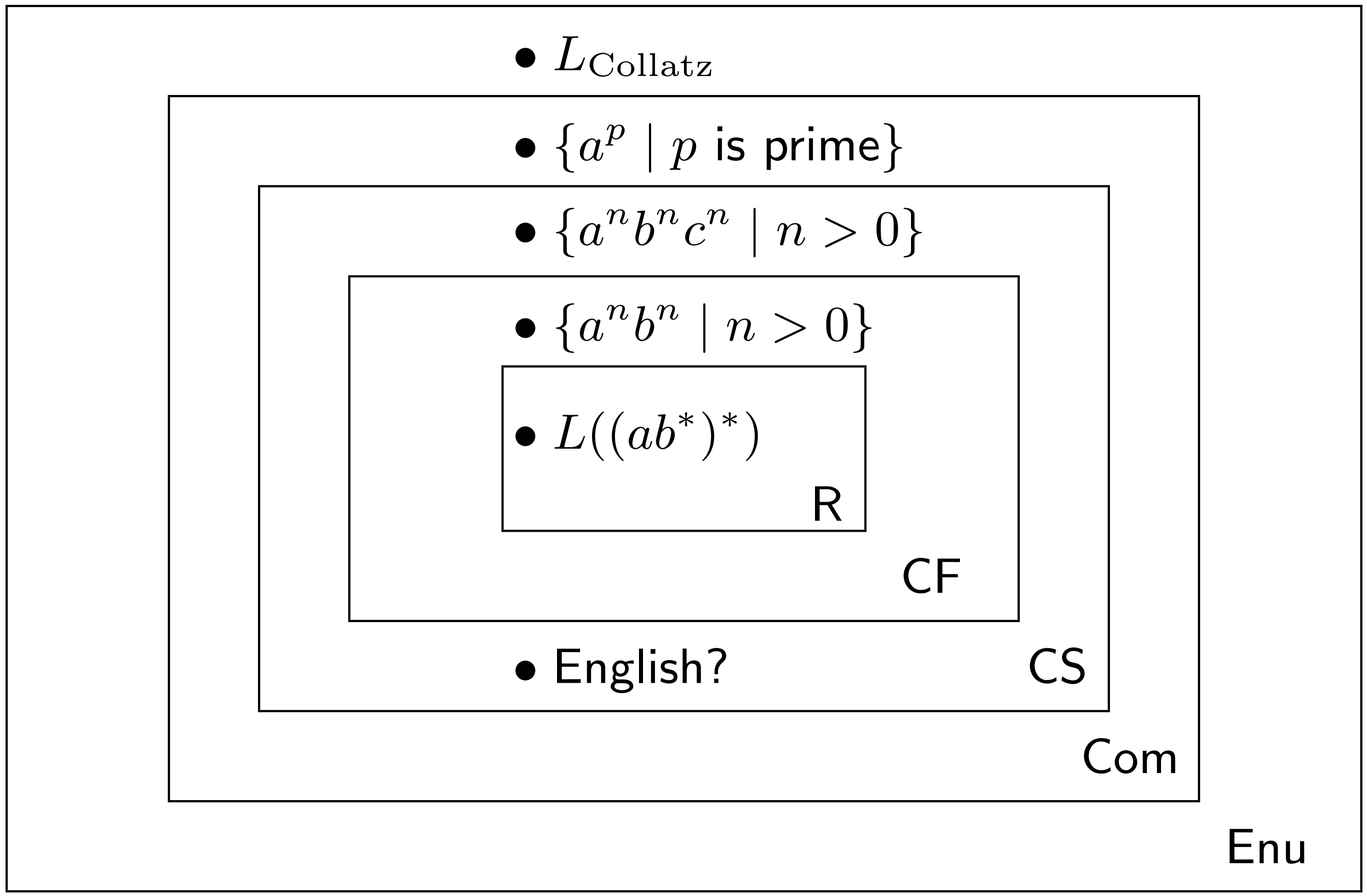
For *unrestricted languages* the production rules are of the form

$$u \to v,$$

with $u, v \in (\Sigma \cup V)^*$.

These are equivalent to the enumerable languages.

A languages is called *computable* if both L and $\overline{L} = \Sigma^* - L$ are enumerable. Let R, CF, CS, Com, Enu be notations for the regular, context-free, context-sensitive, computable and enumerable languages, respectively. Then R $\subseteq$ CF $\subseteq$ CS $\subseteq$ Com $\subseteq$ Enu.



The Chomsky hierarchy

Let $\Sigma = \{\mathsf{M}, \mathsf{I}, \mathsf{U}\}$.
Define the language L over Σ by the following grammar (Hofstadter).

axiom	MI
rules	$x\mathsf{I} \Rightarrow x\mathsf{IU}$ $\mathsf{M}x \Rightarrow \mathsf{M}xx$ $x\mathsf{III}y \Rightarrow x\mathsf{U}y$ $x\mathsf{UU}y \Rightarrow xy$

Here $x, y \in \Sigma^*$. This means that by definition $\mathsf{MI} \in L$

if $x\mathsf{I} \in L$, then also $x\mathsf{IU} \in L$

if $\mathsf{M}x \in L$, then also $\mathsf{M}xx$

if $x\mathsf{III}y \in L$, then also $x\mathsf{U}y$

if $x\mathsf{UU}y \in L$, then also xy

Problem. Does MU belong to L?

Define L_{Collatz} as follows.

axiom	a
rule	$w \Rightarrow ww$ $wwwaa \Rightarrow wwa$

Show that $\{a^n \mid 1 \leq n \leq 10\} \subseteq L_{\text{Collatz}}$

Prove or refute Collatz' conjecture

$$L_{\text{Collatz}} = \{a^n \mid n \geq 1\}$$

The first correct solution with convincing proof sent by email before 01.07.2012 earns 1000 €.