The second incompleteness theorem

Once more: the fixed point theorem

Write $\underline{A} \equiv \overline{\ulcorner A \urcorner}$: the numeral of the Gödel number of A

Proposition. Given a predicate $P(x)$, with $\mathrm{FV}(A) = \{x\}$.

Then there exists a sentence A such that

$$\vdash A \leftrightarrow P(\underline{A})$$

Proof. There is a primitive recursive function Sub such that for B with $\mathrm{FV}(B) = \{x\}$

$$\mathsf{Sub}(\ulcorner s \urcorner, \ulcorner B(x) \urcorner) = \ulcorner B(s) \urcorner$$

We can extend PA with a functionsymbol $\underline{\mathsf{Sub}}$ and axioms

$$\underline{\mathsf{Sub}}(\underline{s}, \underline{B(x)}) = \underline{B(s)}$$

Define $Q(x) = P(\underline{\mathsf{Sub}}(x,x))$ and $A = Q(\underline{Q(x)})$. Then

$$\begin{array}{rcl} A & = & Q(\underline{Q(x)}) \quad = \quad P(\underline{\mathsf{Sub}}(\underline{Q(x)}, \underline{Q(x)})) \\ & \leftrightarrow & P(\underline{Q(\underline{Q(x)})}) \quad = \quad P(\underline{A}). \ \blacksquare \end{array}$$

Provability formalized

Define

$$\Box\varphi \equiv \exists x.\overline{\mathrm{Prf}}(x, \underline{\varphi})$$

So '$\Box\varphi$' states 'φ is provable in PA'

This $\Box$ satisfies the following provability conditions (Hilbert-Bernays)

D_1 $\quad \mathsf{PA} \vdash \varphi \;\Rightarrow\; \mathsf{PA} \vdash \Box\varphi$

D_2 $\quad \mathsf{PA} \vdash \Box(\varphi\rightarrow\psi)\rightarrow[\Box\varphi \rightarrow \Box\psi]$

D_3 $\quad \mathsf{PA} \vdash \Box\varphi \rightarrow \Box\Box\varphi$

PA is *consistent* if $\not\vdash \bot$

PA is *ω-consistent'* if $\not\vdash \Box\bot$

the original stronger definition of ω-consistency is

$$\mathsf{PA} \vdash \varphi(\overline{n}) \text{ for all } n \;\Rightarrow\; \mathsf{PA} \not\vdash \exists x.\neg\varphi(x)$$

Show that this definition implies the variant definition.

Gödel sentence (application of the fixed point theorem)

$$\mathsf{PA} \vdash G \leftrightarrow \neg\Box G$$

Then we have the following.

(i) If PA is consistent, then $\nvdash G$

(ii) If PA is ω-consistent, then $\nvdash \neg G$

Proof.

$$
\begin{array}{llll}
\text{(i)} \ \vdash G & \Rightarrow & \vdash \Box G & \text{by } D_1 \\
 & \Rightarrow & \vdash \neg\Box G & \text{by definition of } G \\
 & \Rightarrow & \vdash \bot & \text{contradicting the consistency of PA.} \ \blacksquare
\end{array}
$$

$$
\begin{array}{llll}
\text{(ii)} \ \vdash \neg G & \Rightarrow & \vdash \Box G & \text{by definition of } G \\
 & \Rightarrow & \vdash \Box\neg G & \text{by } D_1 \\
 & \Rightarrow & \vdash \Box\bot & \text{contradicting the } \omega\text{-consistency of PA.} \ \blacksquare
\end{array}
$$

Problem of Henkin

Define by the fixed point theorem H such that

$$\mathsf{PA} \vdash H \leftrightarrow \Box H$$

H states that it is provable

Then Henkin's problem is whether H is in fact provable

M. Löb solved this problem in an ingenious way: yes

$$\mathsf{PA} \vdash H$$

Löb's Theorem

Theorem. If PA $\vdash (\Box\varphi)\to\varphi$, then PA $\vdash \varphi$

Proof. Suppose PA $\vdash (\Box\varphi)\to\varphi$ towards PA $\vdash \varphi$

Using the fixed point theorem there is a ψ such that

$$\mathsf{PA} \vdash \psi \leftrightarrow (\Box\psi\to\varphi)$$

Then

$$\begin{array}{rcll}
\Box\psi & \vdash & \Box(\Box\psi\to\varphi), & \text{by } D_1, D_2 \\
\Box\psi & \vdash & \Box\Box\psi\to\Box\varphi, & \text{by } D_2 \\
\Box\psi & \vdash & \Box\Box\psi, & \text{by } D_3 \\
\Box\psi & \vdash & \Box\varphi, & \text{by modus ponens} \\
\Box\psi & \vdash & \varphi, & \text{by the assumption on } \varphi \\
 & \vdash & \Box\psi \to \varphi, & \text{by } \to\text{-introduction} \\
 & \vdash & \psi, & \text{by the construction of } \psi \\
 & \vdash & \Box\psi, & \text{by } D_1 \\
 & \vdash & \varphi. \blacksquare &
\end{array}$$

Corollary The Henkin sentence is true/provable: PA $\vdash H$.

Proof. Since PA $\vdash H \leftrightarrow \Box H$, hence PA $\vdash \Box H\to H$. Now apply Löb. ■

Gödel's second incompleteness theorem

Let $\mathsf{Con}_{\mathsf{PA}} \equiv \neg\Box\bot$. Suppose PA is consistent.

Then PA $\nvdash \mathsf{Con}_{\mathsf{PA}}$.

Proof. Suppose PA $\vdash \neg\Box\bot$. This means PA $\vdash \Box\bot\to\bot$.

But then by Löb one has PA $\vdash \bot$,

against the assumption of consistency. ■

Exercise. If PA $\vdash G \leftrightarrow \neg\Box G$, then PA $\vdash G \leftrightarrow \mathsf{Con}_{\mathsf{PA}}$

Exercise. Let PA $\vdash G_1 \leftrightarrow \Box\neg G_1$. Show that

PA is consistent $\;\Rightarrow\;$ PA $\nvdash \neg G_1$

PA is ω-consistent $\;\Rightarrow\;$ PA $\nvdash G_1$

Exercise. Follow Rosser's construction of an R

such that if PA is consistent, then PA $\nvdash R$ & PA $\nvdash \neg R$

Homework

Follow the lecture notes to show D_1, D_2, D_3