8. Typical exercises: answers

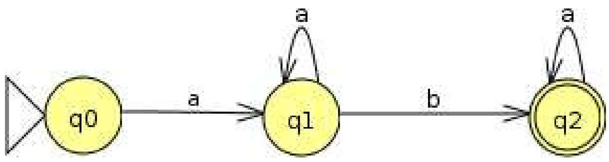
8.1. Given is $\Sigma = \{a, b\}$

Consider

$$L = \{a^n b a^m \mid n > 0,\ m \geq 0\}$$

$$e = aa^* ba^*$$

$M =$



$$G = \boxed{\begin{array}{lcl} S & \to & aA \\ A & \to & aA \mid bB \\ B & \to & aB \mid \lambda \end{array}}$$

Prove that $L = L(e) = L(G) = L(M)$.

Answer. In spite of the hint, we show $L = L(e)$, $L = L(M)$, $L = L(G)$.

$L \subseteq L(e)$. Let $w \in L$, to show $w \in L(e)$. Then $w = a^n b a^m$ for some $n > 0$, $m \geq 0$. Then $n = n' + 1$ and $w = aa^{n'} b a^m \in L(aa^* ba^*) = L(a)L(a^*)L(b)L(a^*)$, as $L(\sigma) = \{\sigma\}$ and $L(a^*) = \{a^k \mid k \geq 0\}$.

$L(e) \subseteq L$. Let $w \in L(e) = L(a)L(a^*)L(b)L(a^*)$. Then

$$w = aa^n ba^m = a^{n+1} ba^m \in L.$$

$L \subseteq L(M)$. Let $w \in L$, to show $w \in L(M)$. Again $w = a^{n+1} ba^m$. Now

$$[q_0, aa^n ba^m] \vdash [q_1, a^n ba^m] \vdash \ldots \vdash [q_1, ba^m] \vdash [q_2, a^m] \vdash \ldots \vdash [q_2, \lambda],$$

hence $w \in L(M)$.

$L(M) \subseteq L$. Let $w \in L(M)$ to show $w \in L$. Then $[q_0, w] \vdash [q_2, \lambda]$, So $w = aw'$ with $[q_1, w'] \vdash [q_2, \lambda]$. But then $w' = a^n bw''$ with $[q_1, bw''] \vdash [q_2, \lambda]$. It follows that $[q_2, w''] \vdash [q_2, \lambda]$, hence $w'' = a^m$. Harvesting:

$$w = aw' = aa^n bw'' = aa^n ba^m = a^{n+1} ba^m \in L.$$

$L \subseteq L(G)$. Take $w \in L$. Then $w = aa^n ba^m$. Now

$$S \Rightarrow aA \Rightarrow aa^n A \Rightarrow aa^n bB \Rightarrow aa^n ba^m,$$

so $w \in L(G)$.

$L(G) \subseteq L$. Claim.
(i) If $B \Rightarrow^* V$, then $V = a^n B$ or $V = a^n$, with $n \geq 0$.
(ii) If $A \Rightarrow^* V$, then $V = a^n A$, $V = a^n ba^m B$, or $V = a^n ba^m$ with $n, m \geq 0$.
(iii) If $S \Rightarrow^* V$, then V is of the form $a^{n+1} A$, $a^{n+1} ba^m B$, $a^{n+1} ba^m$, with $n, m \geq 0$.

We show this in detail for the first implication. Suppose $B \Rightarrow^* V$. By induction on the number of steps in this reduction we get the following.
Possibility 1. $V = B$. Then taking $n = 0$, we have $V = a^n B$.
Possibility 2. $B \Rightarrow aB \Rightarrow^* V$. Then $V = aV'$ and $B \Rightarrow^* V'$ in less steps. By the Induction Hypothesis one has $V' = a^{n'}B$ or $V' = a^{n'}$. Then also $V = aV'$ is of the right form.

Now if $S \Rightarrow^* w \in \Sigma^*$, then by (iii) of the claim $w = a^{n+1}ba^m$ (the other possibilities contain auxiliary symbols), so $w \in L$.

8.2. Given is $\Sigma = \{a, b\}$

Consider

$$L = \{w \in \Sigma^* \mid \#_a(w) = \#_b(w)\}$$

M =

$$G = \boxed{\begin{array}{lcl} S & \to & aB \mid bA \mid \lambda \\ A & \to & aS \mid bAA \\ B & \to & bS \mid aBB \end{array}}$$

(i) Prove that $L = L(G) = L(M)$

(ii) Prove that there is no regular expression e such that $L = L(e)$
Answer.

$L \subseteq L(G)$. Let $w \in L$. Consider the properties

$$\begin{array}{rcl} 1ma(w) & \Leftrightarrow_{\text{Def}} & \#_a(w) = \#_b(w) + 1 \\ 1mb(w) & \Leftrightarrow_{\text{Def}} & \#_b(w) = \#_a(w) + 1 \\ [a{=}b](w) & \Leftrightarrow_{\text{Def}} & \#_b(w) = \#_a(w) \end{array}$$

Claim. For every word w of length n one has:

$$\begin{array}{lccc} (1) & [a{=}b](w) & \Rightarrow & S \Rightarrow^* w; \\ (2) & 1ma(w) & \Rightarrow & A \Rightarrow^* w; \\ (3) & 1mb(w) & \Rightarrow & B \Rightarrow^* w. \end{array}$$

Case $n = 0$. Then $w = \lambda$ and $[a{=}b](\lambda)$. Indeed $S \Rightarrow \lambda$.
Case $n > 0$. Subcase $w = aw'$. If $[a{=}b](w)$, then $1mb(w')$. By the induction hypothesis one has $B \Rightarrow^* w'$. Then $S \Rightarrow aB \Rightarrow^* aw' = w$. If $1ma(w)$, then $[a{=}b](w')$ and hence by the induction hypothesis $S \Rightarrow^* w'$. Then $A \Rightarrow aS \Rightarrow^* aw' = w$. If $1mb(w)$, then $w' = w_1w_2$ with $1mb(w_i)$. By the induction hypothesis $B \Rightarrow^* w_i$. But then $A \Rightarrow aBB \Rightarrow^* aw_1w_2 = aw' = aw$. The subcase $w = bw'$ is treated similarly.

$L(G) \subseteq L$. Following the grammar (by induction on the number of generation steps) we see that

$$\begin{aligned} S \Rightarrow^* w &\quad \Rightarrow \quad [a{=}b](w); \\ A \Rightarrow^* w &\quad \Rightarrow \quad 1ma(w); \\ B \Rightarrow^* w &\quad \Rightarrow \quad 1mb(w). \end{aligned}$$

Therefore if $w \in L(G)$, then $S \Rightarrow^* w$, hence $[a{=}b](w)$, so $w \in L$.
$L \subseteq L(M)$. Let $w \in L$, i.e. $[a{=}b](w)$. Reading this word by M, if the stack is empty, this means that so far the number of a's equals that of b's. If A is on top of the stack and we read a b, then the A, witnessing a previously read a is cancelled while reading the b.

The exercises on the exam will be simpler!
.