Week 5. Simple type theory

In-class problems

1. Find types for the following terms:

   - $\mathtt{I} = \lambda x.x$
   - $\mathbf{1} = \lambda xy.xy$
   - $\mathtt{c}_{\mathtt{exp}} = \lambda xy.yx$
   - $\mathtt{S} = \lambda xyz.xz(yz)$
   - $\mathtt{B} = \lambda fgx.f(gx)$
   - $\lambda xy.x(yx)$

2. Find terms having the following types:

   - $a \to b \to b$
   - $(a \to a \to b) \to a \to b$
   - $(a \to b \to c) \to b \to a \to c$

3. For a type a, let $\neg a$ denote the type $a \to o$. Find terms having the following types:

   (a) $a \to \neg\neg a$
   (b) $(a \to b) \to \neg b \to \neg a$
   (c) $\neg\neg\neg a \to \neg a$
   (d) $(o \to a) \to \neg\neg(\neg\neg a \to a)$

4. Give a proof in minimal propositional logic of the formula

   $$((((a \to b) \to a) \to a) \to b) \to b$$

   Give the proof term that corresponds to this proof in the simply typed lambda calculus and give its type derivation in that system.

5. Prove that there are no closed terms of type o. You may assume

   Subject Reduction: If $\Gamma \vdash M : A$ and $M \twoheadrightarrow N$, then $\Gamma \vdash N : A$.

   Strong Normalization: If $\Gamma \vdash M : A$, then every reduction sequence starting from M is finite. Hence M has a normal form $\mathsf{nf}(M)$.

Take-home problems

1. Suppose that $\vdash M : A$ and $\vdash N : B$.
   (a) Does the pair $\langle M, N \rangle$ have a type?
   (b) What types must the projections π_1, π_2 have in order that $\vdash \pi_1\langle M, N \rangle : A$ and $\vdash \pi_2\langle M, N \rangle : B$?

2. Prove that the set of normal inhabitants of the type

$$N = o \to (o \to o) \to o$$

   is precisely the set $\{\mathsf{c}_n \mid n \in \mathbb{N}\}$

3. Explain why there is no term Y of the simply typed lambda calculus such that

$$Y F \twoheadrightarrow_\beta F\,(Y F)$$

4. (a) Give a cut-free proof in sequent calculus of

$$((a \to b) \to a) \to a$$

   This is called *Peirce's law*.
   (b) Show that there is no proof of Peirce's law in minimal propositional logic.

5. Show that for any polynomial with non-negative integer coefficients

$$p(n) = a_n x^n + \cdots + a_1 x + a_0$$

   (so $a_i \in \mathbb{N}$), there exists a term P of type $N \to N$ which λ-defines $p(n)$.