Lambda Calculus

Week 14

The system λ^*

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The system λ^*: psudo-expressions

Pseudo-expressions $\boxed{\mathcal{T} = V \mid \mathcal{T}\mathcal{T} \mid \lambda V{:}\mathcal{T}.\mathcal{T} \mid \Pi V{:}\mathcal{T}.\mathcal{T}}$

where V is an infinite collection of variables

Reduction on $\mathcal{T}$ generated by $(\lambda x{:}A.B)C \to_\beta B[x := C]$

Statements are of the form $A : B$, with $A, B{\in}\mathcal{T}$

Declarations are of the form $x{:}A$, with $A{\in}\mathcal{T}$ (predicate) and $x{\in}V$ (subject)

Pseudo-context: a finite ordered sequence of declarations all with distinct subjects

The empty context is denoted by $<>$

If $\Gamma =< x_1{:}A_1, \ldots, x_n{:}A_n >$, then

$$\Gamma, x{:}B =< x_1{:}A_1, \ldots, x_n{:}A_n, x{:}B > .$$

Usually we do not write the $<>$

Axiomatization of the notion

$$\Gamma \vdash A : B$$

stating that $A : B$ can be derived from the pseudo-context Γ

in that case A and B are called (legal) expressions and Γ is a (legal) context.

The system λ^*: axioms and rules

Remember

$$A{\to}B \triangleq \Pi x{:}A.B$$

for fresh x

Bases have to become linearly ordered. The reason is that in one wants to derive

$$\begin{array}{rcl} \alpha{:}*, x{:}\alpha & \vdash & x{:}\alpha; \\ \alpha{:}* & \vdash & (\lambda x{:}\alpha.x) : (\alpha \to \alpha) \end{array}$$

but not

$$\begin{array}{rcl} x{:}\alpha, \alpha{:}* & \vdash & x : \alpha; \\ x{:}\alpha & \vdash & (\lambda\alpha{:}*.x) : (\Pi\alpha{:}*.\alpha) \end{array}$$

The notion of type derivation (to come) $\Gamma \vdash A : B$ is defined by the following axioms and rules (the proviso in the conversion rule $(A =_\beta B)$ is a priori not decidable; however it can be replaced by the decidable condition

$$A \to_\beta B \text{ or } B \to_\beta A$$

without changing the set of derivable statements)

The system λ^*: axioms and rules

(axioms)	$\vdash * : *$	
(start)	$\dfrac{\Gamma \vdash A : *}{\Gamma, x : A \vdash x : A}$	if $x \notin \Gamma$
(weakening)	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : *}{\Gamma, x : B \vdash M : A}$	if $x \notin \Gamma$
(product)	$\dfrac{\Gamma \vdash A : * \quad \Gamma, x{:}A \vdash B : *}{\Gamma \vdash (\Pi x{:}A.B) : *}$	
(application)	$\dfrac{\Gamma \vdash F : (\Pi x{:}A.B) \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]}$	
(abstraction)	$\dfrac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.B) : *}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)}$	
(conversion)	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : * \quad A =_\beta B}{\Gamma \vdash M : B}$	

$$A, B, a, b, F, M \in \mathcal{T}$$

Second-order definability of the logical operations

1. For $A, B{:}*$ define

$$
\begin{array}{rcl}
\bot & \triangleq & (\Pi\beta{:}*.\beta) \\
\neg A & \triangleq & (A \to \bot) \\
A \& B & \triangleq & \Pi\gamma{:}*.(A \to B \to \gamma) \to \gamma \\
A \vee B & \triangleq & \Pi\gamma{:}*.(A \to \gamma) \to (B \to \gamma) \to \gamma
\end{array}
$$

2. For $A{:}*$ and $S{:}*$ define

$$
\begin{array}{rcl}
\forall x{:}S.A & \triangleq & \Pi x{:}S.A \\
\exists x{:}S.A & \triangleq & \Pi\gamma{:}*.(\Pi x{:}S.(A \to \gamma)) \to \gamma
\end{array}
$$

3. For $S{:}*$ and $x, y{:}S$ define

$$
(x =_L y) \quad \triangleq \quad \Pi P{:}(S \to *).Px \to Py
$$

Russell paradox

Russell. Let $R = \{a \mid a \notin a\}$. Then

$$\forall a[a \in R \iff a \notin a]$$

In particular

$$R \in R \iff R \notin R$$

Positive version Proposition 1. Let $R = \{a \in X \mid a \notin a\}$. Then

$$R \notin A$$

Hypergame paradox

Zwicker. A game for two players is finite if each game has to finish.

For example tic, tac, toe is finite, but chess not. *Hypergame* is a game for two in which the first player chooses a finite game. After that the second player makes the first move in that game and the first player the second. Etcetera.

Clearly hypergame is finite: as soon as the first player has chosen a finite game, only finitely many moves are possible.

But now starting hypergame the following stream of moves is possible:

hypergame, hypergame, hypergame, ...

which does not finish. Therefore hypergame is not finite. Contradiction.

Positive version Proposition 2. Let R be a binary relation on a set A. Define for $a{\in}A$

$$SN_R a \quad \Leftrightarrow \quad \text{there is no infinite sequence } a_0, a_1, \ldots \in A \text{ such that } \ldots\ldots R a_1 R a_0 R a.$$

Then in A we have

$$\neg\exists b \forall a\, [SN_R a \leftrightarrow aRb].$$

Exercise.

Taking for R the relation in on the universe of sets V gives the Mirimanoff paradox

Quine Proposition 3. Let R be a binary relation on a set A.

For $n = 1, 2, \ldots, \infty$ define

$$C_n a \quad \Leftrightarrow \quad \exists a_0, \ldots, a_n \in A[a_0 = a \;\&\; \forall i < n \; a_{i+1} R a_i \;\&\; a_n = a].$$

$$B_n = \{a \in A \mid \neg C_n a\}.$$

$\{$The set B_n consists of those $a \in A$ not on an 'n-cycle'$\}$. Then in A one has

$$\neg \exists b \forall a [B_n a \leftrightarrow aRb].$$

Proof. Exercise. ■

For $n = 1$ we get the positive version of the Russell paradox.

For $n = \infty$ of the second paradox.

For $n = 2$ we get the 'exclusive club' by Smullyan. (A person is a member of this club if and only if he does not shave anyone who shaves him. Show that there is no person that has shaved every member of the exclusive club and no one else.)

Towards formalization 1

Now we will define a (naive) set T with a binary relation $<$ on it such that

$$\forall a{\in}T\ [SN_{<}a \quad \leftrightarrow \quad a < b], \tag{!}$$

for some $b{\in}T$. Together with Proposition 2 this gives the paradox. The particular choice of T and $<$ is such that the auxiliary lemmas needed can be formalized in λ^*.

Definition 4

(i) $T = \{(A,R) \mid R$ be is a binary transitive relation on set $A\}$

For $(A,R),(A',R'){\in}T$ and $f{:}A \to A'$ write

(a) $(A,R) <_f^- (A',R') \quad \Leftrightarrow \quad \forall a,b{\in}A\ [aRb \quad \to \quad f(a)R'f(b)]$;
(b) f is bounded $\quad \Leftrightarrow \quad \exists a'{\in}A' \forall a{\in}A. f(a)R'a'$;
(c) $(A,R) <_f (A',R') \quad \Leftrightarrow \quad (A,R) <_f^- (A',R')\ \&\ f$ is bounded.

(ii) Define the binary relation $<$ on T by

$$(A,R){<}(A',R') \quad \Leftrightarrow \quad \exists f{:}(A \to A')[(A,R) <_f (A',R')].$$

(iii) Let $W = \{(A,R){\in}T \mid SN_{<}(A,R)\}$.

We will see that $b = (W,<){\in}T$ satisfies (!) above

(For notational simplicity we also write $<$ for the restriction of $<$ to W)

Towards formalization 2

Definition 5. For $(A, R) \in T$ and $a \in A$ write

1. $A_a = \{b \in A \mid bRa\}$;
2. R_a is the restriction of R to A_a.

Lemma 6. Let $(A, R) \in T$ and $a, b \in A$. Then

1. $(A_a, R_a) < (A, R)$;
2. $aRb \;\; \to \;\; (A_a, R_a) < (A_b, R_b)$;
3. $aRb \;\; \to \;\; SN_R b \;\; \to \;\; SN_R a$;
4. $[\forall a \in A SN_R a] \;\; \to \;\; SN_<(A, R)$.

Definition 7. For $(A, R) \in T$ and $a \in A$ write

1. $A_a = \{b \in A \mid bRa\}$;
2. R_a is the restriction of R to A_a.

Towards formalization 3

Lemma 8. Let $(A, R) \in T$ and $a, b \in A$. Then

1. $(A_a, R_a) < (A, R)$;
2. $aRb \;\; \to \;\; (A_a, R_a) < (A_b, R_b)$;
3. $aRb \;\; \to \;\; SN_R b \;\; \to \;\; SN_R a$;
4. $[\forall a \in A SN_R a] \;\; \to \;\; SN_<(A, R)$.

Proof. It suffices to show that for $(A, R) \in T$

1. $SN_<(A, R) \;\; \to \;\; (A, R) < (W, <)$;
2. $SN_<(W, <)$.

For then $(A, R) < (W, <) \;\; \to \;\; SN_<(A, R)$ by Lemma 6(3).

As to 1, suppose $SN_<(A, R)$. Let $a \in A$ and define $f(a) = (A_a, R_a)$, with R_a defined in Def. 7. By Lemma 8 (1) one has $f(a) < (A, R)$; by assumption and 8(3) applied to $(T, <)$ it follows that $SN_<(f(a))$ and hence $f(a) \in W$. Therefore $f{:}A \to W$. Moreover, $f{:}(A, R) < (W, <)$ by Lemma 8(1),(2).

As to 2, note that by definition $\forall (A, R) \in W \; SN_<(A, R)$. Hence by Lemma 8(4) one has $SN_<(W, <)$. ■

Definition 9.

1. Γ_0 is the context
$$A{:}*, R{:}(A \to A \to *).$$

2. Write in context Γ_0
$\mathsf{chain}_{A,R} \triangleq \lambda P{:}(A \to *).\forall a{:}A[Pa \to \exists b{:}A[Pb \mathbin{\&} bRa]]$
$SN_{A,R} \triangleq \lambda a{:}A.\forall P{:}(A \to *)[\mathsf{chain}_{A,R}P \ \to \ \neg Pa]$.

Intuitively, $\mathsf{chain}_{A,R}P$ states that $P{:}A \to *$ is a predicate on (subset of) A such that for every element a in P there is an element b in P with bRa. Moreover $\mathsf{SN}_{A,R}a$ states that $a{:}A$ is not in a subset $P \subseteq A$ that is a chain.

Lemma 10. In λU one can show

1. $\Gamma_0 \vdash \mathsf{chain}_{A,R} : ((A \to *) \to *)$.
2. $\Gamma_0 \vdash SN_{A,R} : (A \to *)$.

Lemma 11. $\Gamma_0 \vdash_{\lambda *} M : \neg\exists b{:}A\forall a{:}A[SN_{A,R}a \leftrightarrow aRb]$, for some term M.

Definition 12.

1. In context Γ_0 write
   $\mathsf{closed}_{A,R} \equiv\ \lambda Q{:}(A \to *).\forall a, b{:}A\ [Qa \to bRa \to Qb]$.
   $\{\mathsf{closed}_{A,R}Q$ says: 'if a is in Q and b is R-below a, then b is in Q'.$\}$

2. In context $\Gamma_0, Q{:}A \to *$, write (relativizing to a predicate Q)

$$\forall a{:}A^Q.B \equiv \forall a{:}A[Qa \to B]$$

$$\exists a{:}A^Q.B \equiv \exists a{:}A[Qa\&B].$$

Corollary 13. In context $\Gamma_0, Q{:}A \to *$ the type

$$\mathsf{closed}_{A,R}Q \to \neg\exists b{:}A^Q\forall a{:}A^Q\ [SN_{A,R}a \leftrightarrow aRb]$$

is inhabited in λU.

Proof. Relativize the proof of Lemma 11 formalized in λ^*. ∎

Definition 14. In λU define two predicates $<^-, <$ of type

$$[\Pi\alpha{:}*\,\Pi r{:}(\alpha \to \alpha \to *)\Pi\alpha'{:}*\,\Pi r'{:}(\alpha' \to \alpha' \to *)\Pi f{:}(\alpha \to \alpha').*]$$

as follows. We write

$$(A,R)<^-_f(A',R') \text{ for } <^- A R A' R' f$$

and similarly for $<$.

1. $(A,R) <^-_f (A',R') \;\Leftrightarrow\; \forall a,b{:}A\,[aRb \;\to\; (fa)R'(fb)]$.
2. $$\begin{array}{lll} (A,R)<_f(A',R') & \Leftrightarrow & (A,R) <^-_f (A',R')\ \& \\ & & \exists a'{:}A'\ [Dom_{R'}a'\& \\ & & \forall a{:}A\ [Dom_R a \to (fa)R'a']], \end{array}$$

   where $\mathrm{Dom}_R a$ stands for $\exists b{:}A.aRb$.
3. Write for the appropriate A,R and A',R'

$$(A,R)<^-(A',R') \;\Leftrightarrow\; \exists f{:}A \to A'\ (A,R) <^-_f (A',R')$$

   and similarly for $<$.

The notion $SN_<$ is not a particular instance of the notion $SN_{A,R}$ because the 'set'

$$\{(A,R) \mid A{:}{*}, R{:}A \to A \to *\}$$

on which $<$ is supposed to act does not form a type; so $SN_<$ has to be defined separately.

Definition 15.

1. $\mathsf{chain}_< \equiv \lambda P{:}(\Pi\alpha{:}*.(\alpha \to \alpha \to *) \to *).$
   $[\forall\alpha_1{:}*\,\forall r_1{:}(\alpha_1 \to \alpha_1 \to *).$
   $[P\alpha_1 r_1 \to \exists\alpha_2{:}*\,\exists r_2{:}(\alpha_2 \to \alpha_2 \to *)$
   $[P\alpha_2 r_2 \& (\alpha_2, r_2) < (\alpha_1, r_1)]$
   $]$
   $].$
2. $SN_< \equiv \lambda\alpha{:}*\,\lambda r{:}(\alpha \to \alpha \to *).\forall P{:}[\Pi\alpha'{:}*.(\alpha' \to \alpha' \to *) \to *].$
   $[\mathsf{chain}_< P \to \neg(P\alpha r)].$
3. Trans $\mathsf{Trans} \equiv \lambda\alpha{:}*\,\lambda r{:}(\alpha \to \alpha \to *).\forall a,b,c{:}\alpha.[arb \to brc \to arc].$
4. In context $\Gamma_0, a{:}A$ define

$$R^a \equiv \lambda b,c{:}A.[bRc \& bRa].$$

Proposition 16. In context

$$A{:}*, R{:}(A \to A \to *), a{:}A, b{:}A, x : \mathsf{Trans}AR$$

the following types are inhabited.

1. $\mathsf{Dom}_R a \ \to \ (A, R^a) < (A, R)$.
2. $aRb \ \to \ (A, R^a) < (A, R^b)$.
3. $aRb \ \to \ SN_{A,R}b \ \to \ SN_{A,R}a$.
4. $(\forall a{:}A.SN_{A,R}a) \ \leftrightarrow \ SN_< AR$.

A universal notation system in $\lambda *$

Now the second problem mentioned in Step 3 will be solved. Terms $\mathbf{U}$ and $\mathbf{i}$ will be constructed such that $\mathbf{i}$ faithfully embeds a pair (A, R) with $A{:}n$ and $R{:}(A \to A \to *)$ into $\mathbf{U}$. Such a pair $(\mathbf{U}, \mathbf{i})$ is called a *universal notation system* for orderings and plays the role of the naive set $T = \{(A, R) \mid R{:}A \to A \to *\}$.

Definition 17.
$$\begin{array}{rcl} H & \equiv & \Pi\alpha{:}*.[(\alpha \to \alpha \to *) \to *]; \\ \mathbf{U} & \equiv & H \to *; \\ \mathbf{i} & \equiv & \lambda\alpha{:}*\,\lambda r{:}(\alpha \to \alpha \to *)\lambda h{:}H.h\alpha r. \end{array}$$

Proposition 18.

1. $\vdash \mathbf{U} : *$.
2. $\vdash \mathbf{i} : [\Pi\alpha{:}*.(\alpha \to \alpha \to *) \to \mathbf{U}]$.
3. The type {'faithfulness of the map $\mathbf{i}$'}

$$\forall\alpha{:}*\,\forall r{:}(\alpha \to \alpha \to *)\forall\alpha'{:}*\,\forall r'{:}$$

$$(\alpha' \to \alpha' \to *)[\mathbf{i}\alpha r =_L \mathbf{i}\alpha' r' \to (\alpha', r) <^- (\alpha', r')]$$

is inhabited.

Proof of Proposition 18

Clearly one has in λU

$$H : *, \mathbf{U} : * \text{ and } \mathbf{i} : [\Pi\alpha{:}\, * .(\alpha \to \alpha \to *) \to (H \to *)].$$

So we have 1 and 2. As to 3, we must show that in context

$$\alpha{:}*, r{:}(\alpha \to \alpha \to *), \alpha'{:}*, r'{:}(\alpha' \to \alpha' \to *)$$

the type

$$\mathbf{i}\alpha r =_L \mathbf{i}\alpha' r' \to (\alpha, r) <^- (\alpha', r')$$

is inhabited. Now

$$\begin{array}{ll} & \mathbf{i}\alpha r =_L \mathbf{i}\alpha' r' \\ \to & \lambda h{:}H.h\alpha r =_L \lambda h{:}H.h\alpha' r' \\ \to & h\alpha r =_L h\alpha' r', \text{ for all } h{:}H, \\ \to & [(\alpha, r) <^- (\alpha', r')] =_L [(\alpha', r') <^- (\alpha', r')], \end{array}$$

by taking $h \equiv \lambda\beta{:}\, * \,\lambda s{:}(\beta \to \beta \to *).(\beta, s) <^- (\alpha', r')$.

Since the right-hand side of the last equation is inhabited it follows that

$$(\alpha, r) <^- (\alpha', r'). \blacksquare$$

Using $\mathbf{U}$ in $\mathbf{i}$ we now can formalize the informal paradox

0.1. DEFINITION. 1. On $\mathbf{U}$ define the binary relation $<_{\mathbf{i}}$ as follows. For $u, u'{:}\mathbf{U}$ let

$$
\begin{aligned}
u <_{\mathbf{i}} u' \equiv \quad & \exists \alpha{:}* \, \exists r{:}(\alpha \to \alpha \to *) \exists \alpha'{:}* \, \exists r'{:}(\alpha' \to \alpha' \to *). \\
& [u =_L (\mathbf{i}\alpha r) \& u' =_L (\mathbf{i}\alpha' r') \& \\
& \text{Trans } \alpha r \ \& \ \text{Trans } \alpha' r' \ \& \\
& SN_{<}(\alpha, r) \ \& \\
& SN_{<}(\alpha', r') \ \& \ (\alpha, r) < (\alpha', r')].
\end{aligned}
$$

2. On $\mathbf{U}$ define the (unary) predicate $\mathbf{I}$ as follows. For $u{:}\mathbf{U}$ let

$$
\begin{aligned}
\mathbf{I}u = \quad & \exists \alpha{:}* \, \exists r{:}(\alpha \to \alpha \to *). \\
& [u =_L (\mathbf{i}\alpha r) \ \& \ \text{Trans } \alpha r \ \& \ SN_{<}(\alpha, r)].
\end{aligned}
$$

   Note that $\text{closed}_{\mathbf{U},<_{\mathbf{i}}} \mathbf{I}$.

3. The element $\mathbf{u} : \mathbf{U}$ is defined by $\mathbf{u} \equiv \mathbf{i}\mathbf{U} <_{\mathbf{i}}$.

Lemma. The following types are inhabited in context

$$\alpha{:}*, r{:}(\alpha \to \alpha \to *), \alpha'{:}*, r'{:}(\alpha' \to \alpha' \to *)$$

1. $(\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\alpha' r') \ \to \ (\alpha, r) < (\alpha', r')$.
2. $SN_{<}(A, R) \ \to \ SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}AR)$.

0.2. Lemma. *In context* $\alpha{:}*, r{:}(\alpha \to \alpha \to *), \alpha'{:}*, r'{:}(\alpha' \to \alpha' \to *)$ *the following types are inhabited.*

1. $(\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\alpha' r') \ \to \ (\alpha, r) < (\alpha', r')$.

2. $SN_{<}(A, R) \ \to \ SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}AR)$.

Proof. 1. Suppose $(\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\alpha' r')$. Then there are β, s, β', s' of appropriate type such that

$$\mathbf{i}\alpha r =_L \mathbf{i}\beta s \ \& \ \mathbf{i}\alpha' r' =_L \mathbf{i}\beta' s' \ \& \ (\beta, s) < (\beta', s').$$

By the faithfulness of $\mathbf{i}$ and the symmetry of $=_L$ it follows that

$$(\alpha, r) <^{-} (\beta, s) < (\beta', s') <^{-} (\alpha', r')$$

hence

$$(\alpha, r) < (\alpha', r').$$

2. Suppose $SN_{<}(A, R)$. If $\text{chain}_{\mathbf{U},<_{\mathbf{i}}} Q$, then define

$$P\alpha r \ \equiv \ Q(\mathbf{i}\alpha r).$$

Then $\text{chain}_{<} P$. Since $SN_{<}(A, R)$ we have $\neg PAR$. But then $\neg Q(\mathbf{i}AR)$. So we proved

$$\text{chain}_{\mathbf{U},<_{\mathbf{i}}} Q \ \to \ \neg Q(\mathbf{i}AR),$$

i.e. $SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}AR)$. ■

0.3. Lemma. *In context $\alpha{:}*, r{:}(\alpha \to \alpha \to *), \alpha'{:}*, r'{:}(\alpha' \to \alpha' \to *)$ the following types are inhabited.*

1. $(\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\alpha' r') \;\to\; (\alpha, r) < (\alpha', r')$.
2. $SN_{<}(A, R) \;\to\; SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}AR)$.

Proof. 1. Suppose $(\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\alpha' r')$. Then there are β, s, β', s' of appropriate type such that

$$\mathbf{i}\alpha r =_L \mathbf{i}\beta s \;\&\; \mathbf{i}\alpha' r' =_L \mathbf{i}\beta' s' \;\&\; (\beta, s) < (\beta', s').$$

By the faithfulness of $\mathbf{i}$ and the symmetry of $=_L$ it follows that

$$(\alpha, r) <^{-} (\beta, s) < (\beta', s') <^{-} (\alpha', r')$$

hence

$$(\alpha, r) < (\alpha', r').$$

2. Suppose $SN_{<}(A, R)$. If $\text{chain}_{\mathbf{U},<_{\mathbf{i}}} Q$, then define

$$P\alpha r \;\equiv\; Q(\mathbf{i}\alpha r).$$

Then $\text{chain}_{<} P$. Since $SN_{<}(A, R)$ we have $\neg PAR$. But then $\neg Q(\mathbf{i}AR)$. So we proved

$$\text{chain}_{\mathbf{U},<_{\mathbf{i}}} Q \;\to\; \neg Q(\mathbf{i}AR),$$

i.e. $SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}AR)$. ■

0.4. Corollary. *The type*

$$\forall u{:}\mathbf{U}.SN_{\mathbf{U},<_{\mathbf{i}}} u$$

is inhabited.

Proof. Let $u{:}\mathbf{U}$ and suppose towards a contradiction

$$\mathsf{chain}_{\mathbf{U},<_{\mathbf{i}}} P \;\&\; Pu.$$

Then

$$\exists u'{:}\mathbf{U}.(u' <_{\mathbf{i}} u \;\&\; Pu').$$

Now

$$u' <_{\mathbf{i}} u \to \exists \alpha{:}*\, \exists r{:}(\alpha \to \alpha \to *)[u =_L (\mathbf{i}\alpha r) \;\&\; SN_{<}(\alpha, r)].$$

Hence by (2) of the lemma

$$SN_{\mathbf{U},<_{\mathbf{i}}}(\mathbf{i}\alpha r) =_L SN_{\mathbf{U},<_{\mathbf{i}}} u.$$

But then, again using $\mathsf{chain}_{\mathbf{U},<_{\mathbf{i}}} P$, it follows that $\neg(Pu)$. Contradiction. ■

0.5. Lemma. *Let $A{:}*, R{:}(A \to A \to *), x{:}\mathrm{Trans} A\, R$. The following type is inhabited*

$$SN_{<}(A, R) \to \forall a{:}A.SN_{<}(A, R^a).$$

Proof. Applying **??**(4) one has

$$\begin{array}{rcll} SN_{<}(A, R) & \to & \forall b{:}A.SN_{A,R} b, & \\ & \to & \forall b{:}A.\forall a{:}A.SN_{A,(R^a)} b, & \text{see below,} \\ & \to & \forall a{:}A.SN_{<}(A, R^a). & \end{array}$$

The implication $SN_{A,R} b \to SN_{A,(R^a)} b$ is proved as follows. Let $SN_{A,R} b$ and assume towards a contradiction that $\mathsf{chain}_{A,(R^a)} P$ and Pb. Then also $\mathsf{chain}_{A,R} P$, contradicting $\mathsf{SN}_{A,R} b$. ■

0.6. Lemma. *1. Let $\alpha{:}*$ and $r{:}\alpha \to \alpha \to *$ and assume* $\mathrm{Trans}\alpha r$ *&* $SN_<(\alpha, r)$. *Then there are $\alpha^+{:}*$ and $r^+{:}\alpha^+ \to \alpha^+ \to *$ such that*

$$\mathrm{Trans}\ \alpha^+ r^+ \ \&\ SN_<(\alpha^+, r^+)\ \&\ (\alpha, r) < (\alpha^+, r^+).$$

2. $\forall v{:}\mathbf{U^I}\exists v^+{:}\mathbf{U^I} v <_{\mathbf{i}} v^+.$

Proof. 1. The construction is the one for representing data structures in Section **??**. Define

$$\alpha^{\circ} \equiv \Pi\beta{:}*.\beta \to (\alpha \to \beta) \to \beta,$$

$$F \equiv \lambda x{:}\alpha\lambda\beta{:}*\lambda\infty{:}\beta\lambda f{:}(\alpha \to \beta).fx,$$

$$\underline{\infty} \equiv \lambda\beta{:}*\lambda\infty{:}\beta\lambda f{:}(\alpha \to \beta).\infty;$$

then $\underline{\infty}{:}\alpha^{\circ}{:}*$ and $F{:}(\alpha \to \alpha^{\circ})$. Intuitively $\alpha^{\circ} = \alpha \cup \{\underline{\infty}\}$ and F is the canonical imbedding. Indeed, F is injective and $\underline{\infty}$ is not in the range of F. In fact, in the given context one has

$$(\lambda a{:}\alpha\lambda b{:}\alpha\lambda p{:}(Fa =_L Fb)\lambda Q{:}(\alpha \to *).p(\lambda x{:}\alpha^{\circ}.x*\bot Q)) \ : \ (\forall a, b{:}\alpha.(Fa =_L Fb \to a =_L b);$$

$$(\lambda a{:}\alpha\lambda p{:}(Fa =_L \underline{\infty}).p(\lambda x{:}\alpha^{\circ}.x*\bot(\lambda a{:}\alpha.T))(\lambda b{:}\bot.b)) \ : \ (\forall a{:}\alpha.Fa \neq_L \underline{\infty});$$

here $T \equiv \bot \to \bot$ stands for 'true' and has $(\lambda b{:}\bot.b)$ as inhabiting proof. Define $r^{\circ}{:}\alpha^{\circ} \to \alpha^{\circ} \to *$ as the canonical extension of r to α^+ making $\underline{\infty}$ larger than the elements of α:

$$r^{\circ} \equiv \lambda x{:}\alpha^{\circ}\lambda y{:}\alpha^{\circ}.[\exists a{:}\alpha\exists b{:}\alpha.rab \ \&\ x =_L Fa \ \&\ y =_L Fb] \vee$$

$$[\exists a{:}\alpha.x =_L Fa \ \&\ y =_L \underline{\infty}].$$

Then Trans $\alpha^{\circ} r^{\circ} \& SN_<(\alpha^{\circ}, r^{\circ})$ and $(\alpha, r) <_{\overline{F}} (\alpha^{\circ}, r^{\circ})$ with bounding element $\underline{\infty}$. This $\underline{\infty}$ is not yet in Dom_{α^+}; but one has $(\alpha, r) <_{F\circ F} (\alpha^{\circ\circ}, r^{\circ\circ})$ with bounding element $F\underline{\infty}$ and therefore one can take $\alpha^+ = \alpha^{\circ\circ}$ and $r^+ = r^{\circ\circ}$.

If $v = \mathbf{i}\alpha r$, then take $v^+ = \mathbf{i}\alpha^+ r^+$. ∎

0.7. Proposition. *The following type is inhabited:*

$$\exists u{:}\mathbf{U}^{\mathbf{I}}\forall v{:}\mathbf{U}^{\mathbf{I}}[SN_{\mathbf{U},<_{\mathbf{i}}}v \quad\leftrightarrow\quad v <_{\mathbf{i}} u].$$

Proof. For u one can take $\mathbf{u} \equiv (\mathbf{i}\mathbf{U} <_{\mathbf{i}})$. In view of Corollary0.4 it is sufficient to show for $v{:}\mathbf{U}$ that {the following types are inhabited}:

1. $\mathbf{I}\mathbf{u}$,
2. $\mathbf{I}v \to v <_{\mathbf{i}} \mathbf{u}$.

As to 1, we know from Corollary 0.4

$$\begin{array}{ll}\forall u{:}\mathbf{U} & SN_{\mathbf{U},<_{\mathbf{i}}}u \\ & \to SN_{<}\mathbf{U} <_{\mathbf{i}}),\ \text{by Proposition ??(4)}, \\ & \to \mathbf{I}(\mathbf{i}\mathbf{U} <_{\mathbf{i}}),\ \text{since clearly Trans}\ \mathbf{U} <_{\mathbf{i}}, \\ & \to \mathbf{I}\mathbf{u}.\end{array}$$

As to 2, assume $\mathbf{I}v$. Then $v =_L (\mathbf{i}\alpha r)$ for some pair α, r with Transαr & $SN_{<}(\alpha, r)$.
Define

$$f \equiv (\lambda a{:}\alpha.(\mathbf{i}\alpha r^{a})) : (\alpha \to \mathbf{U}).$$

Then for all $a{:}\alpha$ with Dom$_r a$ one has

$$fa = (\mathbf{i}\alpha r^{a}) <_{\mathbf{i}} (\mathbf{i}\alpha r) = v,$$

{by 16(1) one has $(\alpha, r^{a}) < (\alpha, r)$; use Lemma 0.5 and the definition of $<_{\mathbf{i}}$} and similarly for all $a, b{:}\alpha$

$$\begin{array}{lll} arb & \to (\alpha, r^{a}) < (\alpha, r^{b}), & \\ & \to \mathbf{i}\alpha r^{a} <_{\mathbf{i}} \mathbf{i}\alpha r^{b}, & SN_{<}(\alpha, r^{a})\ \&\ SN_{<}(\alpha, r^{b})\ \text{since}\ SN_{<}(\alpha, r), \\ & \to fa <_{\mathbf{i}} fb. & \end{array}$$

Therefore $(\alpha, r) <_{f}^{-} (\mathbf{U}, <_{\mathbf{i}})$; f on Dom$_r$ is bounded by v. Since $v <_{\mathbf{i}} v^{+}$ one has Dom$_{<_{\mathbf{i}}} v$. Therefore $(\alpha, r) <_{f} (\mathbf{U}, <_{\mathbf{i}})$ and hence $v =_L (\mathbf{i}\alpha r) <_{\mathbf{i}} (\mathbf{i}\mathbf{U} <_{\mathbf{i}}) = \mathbf{u}$. ∎

0.8. THEOREM (Girard's paradox). *The type $\perp$ is inhabited in λU and hence in $\lambda *$.*

PROOF. Note that Proposition 0.7 is in contradiction with Corollary 13, since $\mathbf{I}$ is closed in $\mathbf{U}, <_{\mathbf{i}}$. This shows that $\perp$ is inhabited in λU, so a fortiori in $\lambda *$. ■