The Incompleteness of Arithmetic

Aristotle (384-322 BC)

- The axiomatic method $\longmapsto$ Euclid's axiomatization of geometry

objects	properties
primitive	axioms
defined	derived

- The quest for logic: try to chart reasoning

  (finished by Frege [1879]; proved complete by Gödel [1930])

- Proof-checking vs theorem proving [??]

Views on Mathematics

“$\vdash A$” stands for “A is provable”

after Aristotle	after Frege
Axioms	Axioms
$\downarrow$ Reasoning	$\downarrow$ Logic
Mathematics	Mathematics

Gödel	(1931)	Mathematics is incomplete	$\nvdash G$ and $\nvdash \neg G$ for some G
		if it is consistent	'p is a proof of A' is decidable
Turing	(1936)	Mathematics is undecidable	$\{A \mid\ \vdash A\}$ non-computable

COROLLARY. There are relatively short statements with very long proofs

Incompleteness of arithmetic

Arithmetic was axiomatized by Dedekind and Peano

DEFINITION. Let T be an axiomatic theory

(i) T is called *consistent* if for no statement A

$$T \vdash A \text{ and } T \vdash \neg A$$

(ii) T is called *incomplete* if for some statement A

$$\text{neither } T \vdash A \text{ nor } T \vdash \neg A$$

THEOREM. (Gödel [1931] as improved by Rosser [1938])

If arithmetic is consistent, then it is incomplete.

Proof sketch

Arithmetic is about numbers (Write $\vdash A$ if A is provabile in PA; e.g. $\vdash$ Prime($\ulcorner 3 \urcorner$))

Everything is a number (Pythagoras; digital era)

Arithmetic is about everything

Arithmetic is about itself (write $\ulcorner A \urcorner$ for the numeral of a sentence A)

Milestone 1. There is a predicate $B(.)$ such that for all sentences A

$$\begin{array}{rcl} \vdash B(\ulcorner A \urcorner) & \Rightarrow & \vdash A \\ \vdash \neg B(\ulcorner A \urcorner) & \Rightarrow & \nvdash A \end{array} \qquad !$$

Milestone 2. Given a predicate $P(.)$ there exists a sentence A fixed point of $P(.)$

$$\vdash A \leftrightarrow P(\ulcorner A \urcorner)$$

Gödel sentence: Let G be the fixed point of $\neg B(.)$. Then

$$\vdash G \leftrightarrow \neg B(\ulcorner G \urcorner)$$

If $\vdash G$, then $\vdash \neg B(\ulcorner G \urcorner)$, hence $\nvdash G$, impossible.

If $\vdash \neg G$, then $\vdash B(\ulcorner G \urcorner)$, hence $\vdash G$; then PA is inconsistent.

Therefore is PA is consistent, then G is neither provable, nor refutable

Fixed Points

Proposition. Given a predicate $P(.) = P(x)$, there exists a sentence A such that

$$\vdash A \leftrightarrow P(\ulcorner A \urcorner)$$

Proof (sketch). Construct a predicate $D(.)$ such that for all $C(.)$ one has

$$\vdash D(\ulcorner C(.) \urcorner) \leftrightarrow C(\ulcorner C(.) \urcorner)$$

Construct a predicate $H(.)$ such that

$$\vdash H(x) \leftrightarrow P(\ulcorner D(\underset{\cdot}{x}) \urcorner) \qquad !!$$

By this we mean that for all $n \in \mathbb{N}$

$$\vdash H(\ulcorner n \urcorner) \leftrightarrow P(\ulcorner D(\ulcorner n \urcorner) \urcorner)$$

Now take $A = D(\ulcorner H(.) \urcorner)$. Then we have

$$\begin{array}{rcll} \vdash A & \leftrightarrow & D(\ulcorner H(.) \urcorner), & \text{by construction of } A, \\ & \leftrightarrow & H(\ulcorner H(.) \urcorner), & \text{by construction of } D, \\ & \leftrightarrow & P(\ulcorner D(\ulcorner H(.) \urcorner) \urcorner), & \text{by construction of } H, \\ & \leftrightarrow & P(\ulcorner A \urcorner), & \text{by construction of } A. \ \blacksquare \end{array}$$

In short: given P we want an A with $A = PA$. Let $DC = CC, HC = G(DC), A = DH$. Then $A = DH = HH = G(DH) = GA$!

Reformulation (strengthening)

DEFINITION. Given a mathematical theory T.

(i) Write for $A, B \in L_T$, statements in the language of T

$$
\begin{array}{rcl}
A \leq_T B & \Leftrightarrow & T \vdash A \rightarrow B \\
A <_T B & \Leftrightarrow & A \leq_T B \;\&\; B \not\leq_T A \\
A =_T B & \Leftrightarrow & A \leq_T B \;\&\; B \leq_T A
\end{array}
$$

(ii) The Lindenbaum Algebra of T is $LA(T) = \langle L_T / =_T, \leq, \rightarrow \rangle$ where

$$
\begin{array}{rcl}
[A] \leq [B] & \Leftrightarrow & A \leq_T B \\
[A] \rightarrow [B] & = & [A \rightarrow B]
\end{array}
$$

Reformulation of the Gödel-Rosser theorem (inspired by Roel de Vrijer)

THEOREM. *The Lindenbaum Algebra of Peano Arithmetic is dense*

$$A < B \;\Rightarrow\; \exists C . A < C \;\&\; C < B$$

Blow to Hilbert

David Hilbert (8th of September 1930):

to the Society of German Scientists and Physicians, in Königsberg

"*Wir dürfen nicht denen glauben, die heute mit philosophischer Miene und überlegenem Tone den Kulturuntergang prophezeien und sich in dem Ignorabimus gefallen. Für uns gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. Statt des törichten Ignorabimus heiße im Gegenteil unsere Losung:*

Wir müssen wissen — wir werden wissen!"

"We must not believe those, who today, with philosophical bearing and deliberative tone, prophesy the fall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion none whatever in natural science. In opposition to the foolish ignorabimus our slogan shall be:

We must know — we will know!"